合肥學院
HEFEI UNIVERSITY

Metaheuristic Optimization

23. Linear Programming

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Outline

website

Section Outline

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- However, the vast majority of industry applications in optimization do not use metaheuristics.
- They use Linear Programming tools such as as CPLEX and GAMS.

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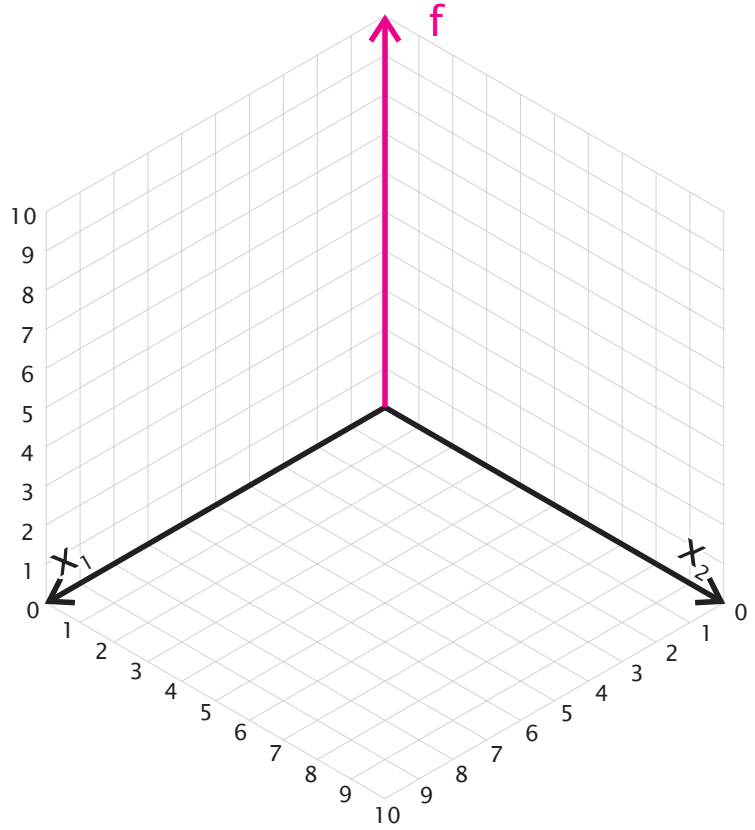
- There are any number m of inequality constraints g_i and any number of equality constraints h_j (see Lesson 16: *Constraint Handling*), each of which are linear.

Example

- Let us define a simple linear programming problem with two decision variables and an objective function.



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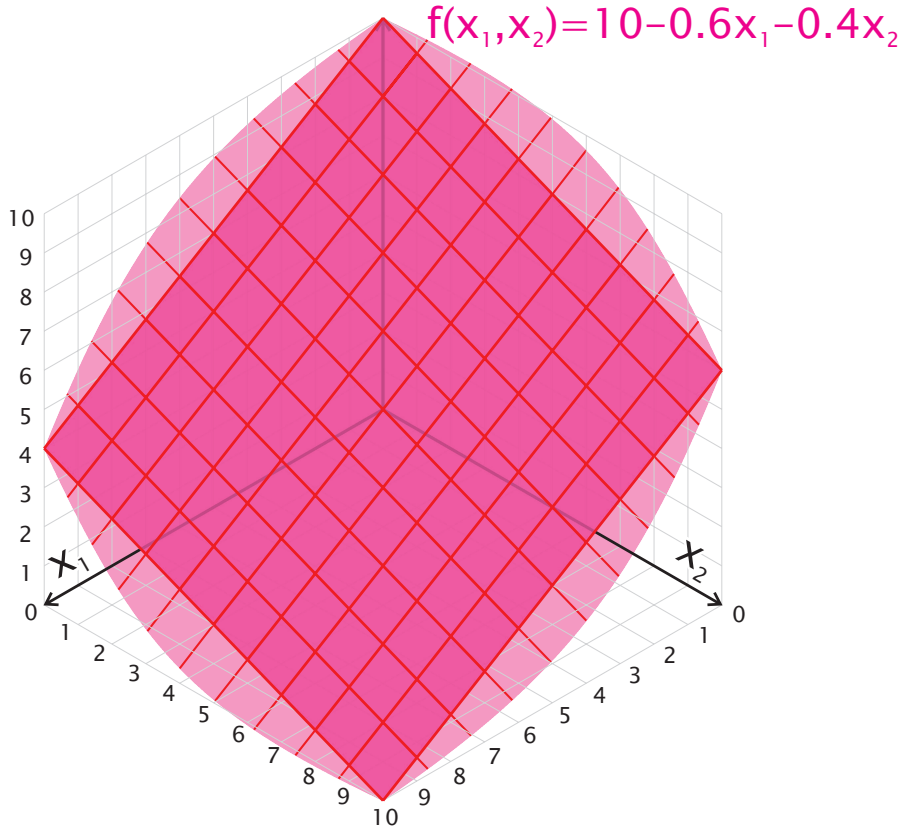

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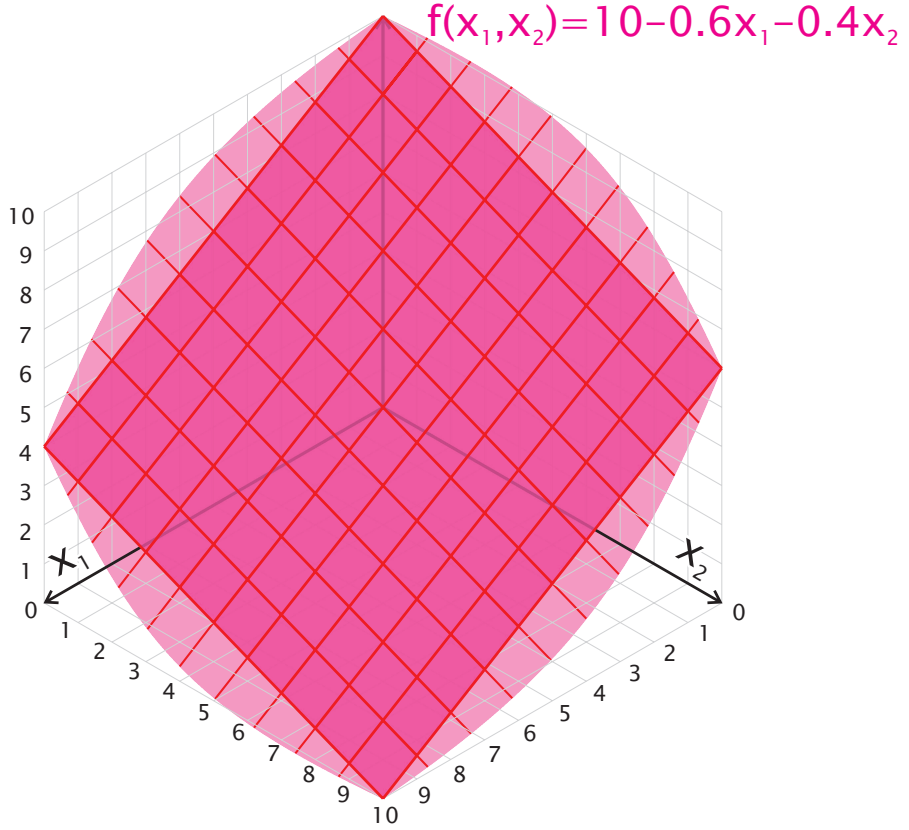

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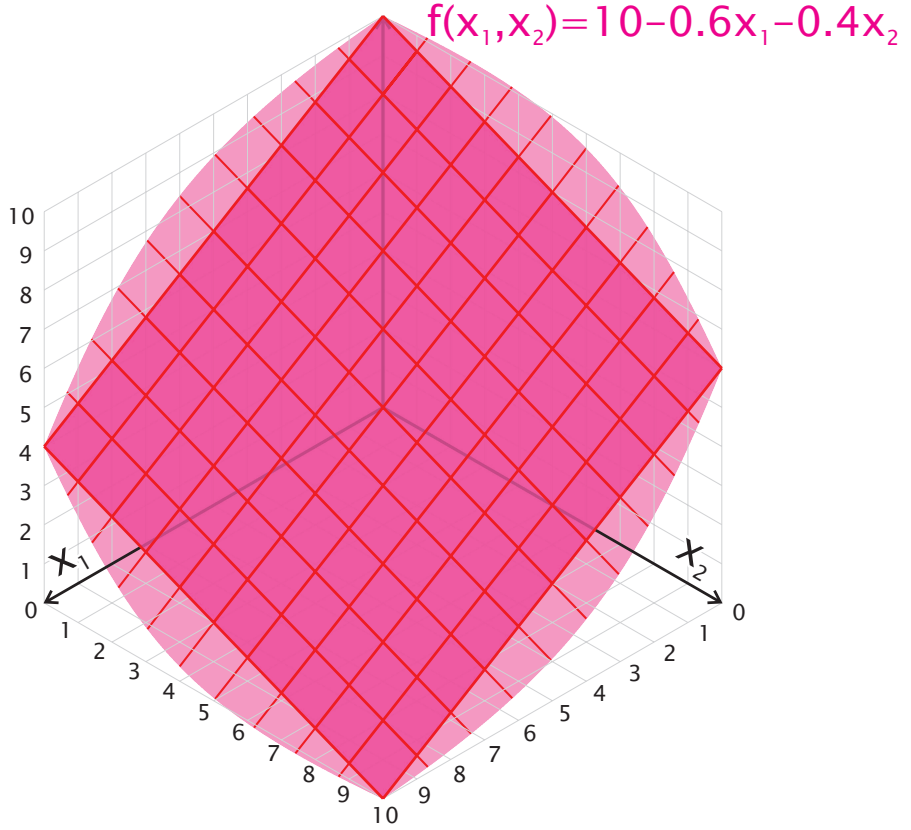

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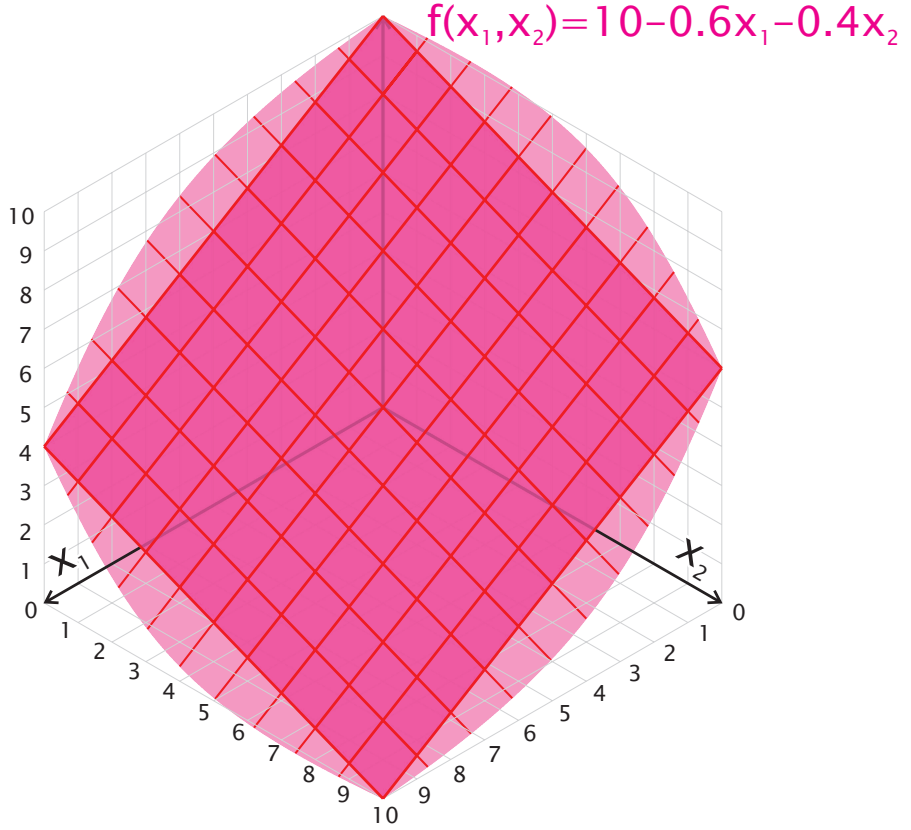

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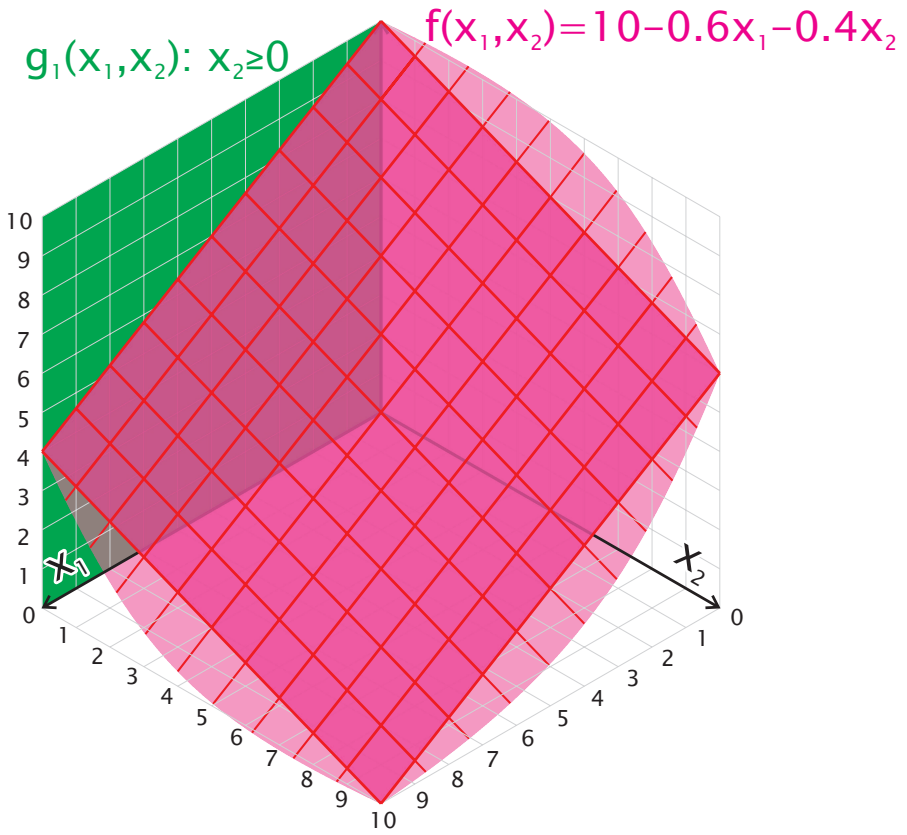

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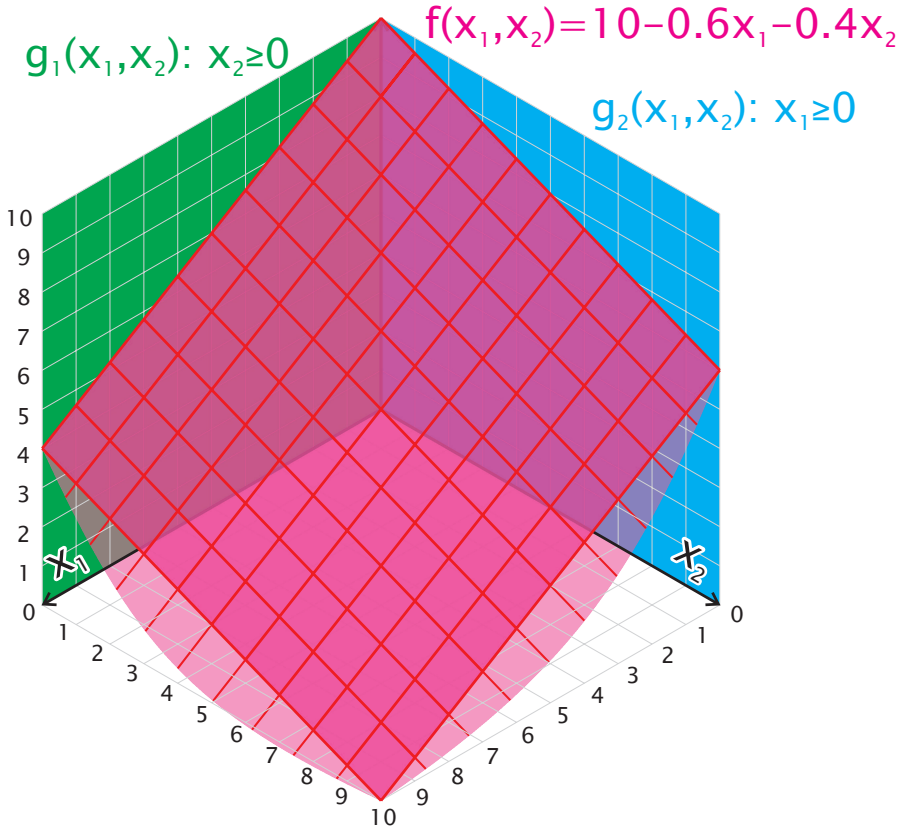

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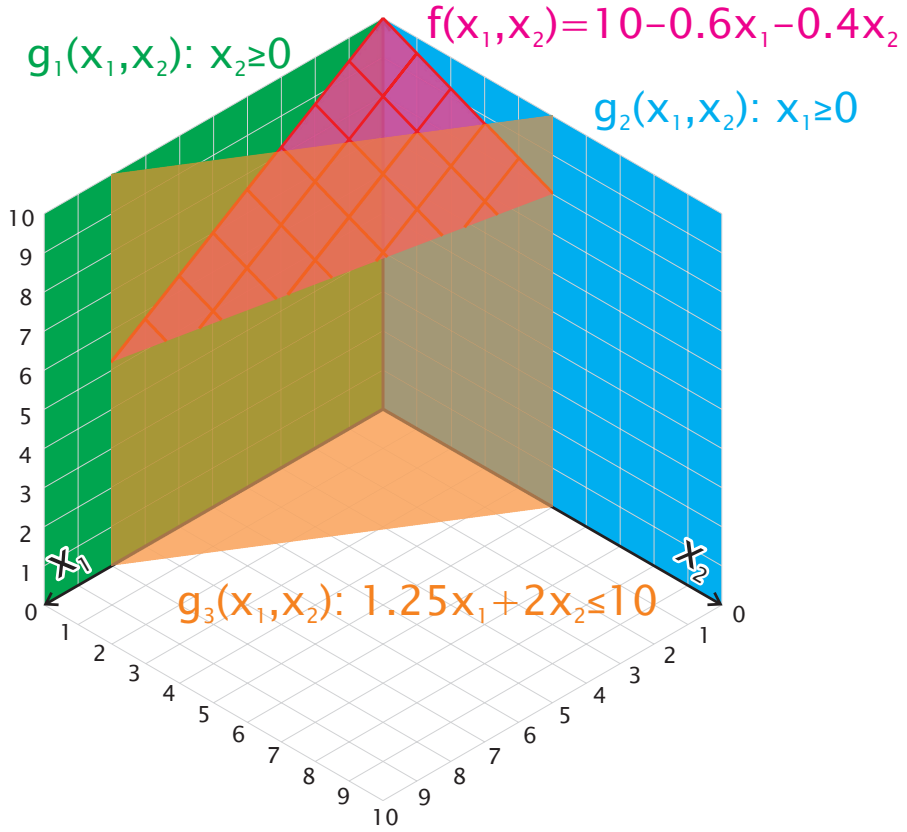

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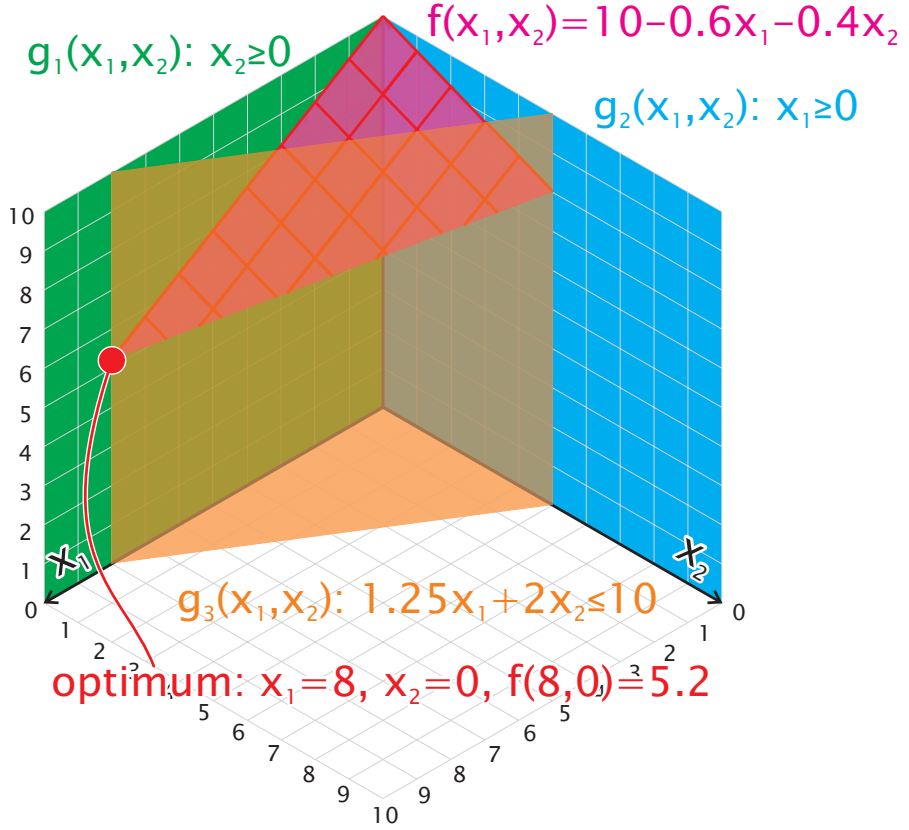
- Furthermore, let's say that $1.25x_1 + 2x_2 \leq 10$ should hold.
- The search space is now limited by linear constraints and within these constraints, the objective function has an optimum with value 5.2 at $x_1 = 8$ and $x_2 = 0$.



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- There are many, many problems in the real-world which fit to this schema
- There are several highly efficient tools for solving such problems to optimality
- If we can bring a problem into this schema, we can apply these tools and get the best possible solution quickly

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$$\textit{minimize} \quad f(\vec{x}) \quad = \quad \vec{c}^T \, \vec{x} \tag{2}$$

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- where $\vec{c}$ is a n-dimensional cost vector, $\cdot^T$ means transposition, $\vec{x}$ the n-dimensional vector of decision variables, A a $m \times n$ matrix and $\vec{b}$ another vector used to specify the constraints, and the "$\geq$" is to be understood as element-wise comparison

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(if we solve the problem using F and get the optimal values for the decision variables x_1 and x_2, we can later compute the value of f)

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- Now we need to translate the $m = 3$ constraints $g_1 : x_2 \geq 0$, $g_2 : x_1 \geq 0$, and $g_3 : 1.25x_1 + 2x_2 \leq 10$ into the form $A\,\vec{x} \geq \vec{b}$

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Section Outline

General Algebraic Modeling System (GAMS)

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- Such problems can be solved using programs such as CPLEX
- For doing so, we can conveniently express them as models, using, for instance the General Algebraic Modeling System (GAMS), which you can download from `http://www.gams.com`

GAMS Example: Definition

- The previous example looks like this in GAMS notation:

Listing: example1.gams

```
Variables f, x1, x2;

Equations obj, g1, g2, g3;

obj.. f =e= 10 - 0.6*x1 - 0.4*x2;
g1..  x2 =g= 0;
g2..  x1 =g= 0;
g3..  1.25*x1 + 2*x2 =l= 10;

Model example1 / obj, g1, g2, g3 /;

solve example1 using LP minimizing f;
```

- where `=e=` means "$=$", `=l=` means "$\leq$", and `=g=` means "$\geq$"

GAMS Example: Execution

- We now apply GAMS:

Listing: Shell command to start GAMS

```
gams example1.gams -o example1.lst -logOption 2
```

- where `-o example1.lst` tells the system to store the output in file `example1.lst`
- and `-logOption 2` stores the console/log output in file `example1.log`
- non-printable characters in `example1.lst` can be removed via `tr -d '000-011013014016-037' < example1.lst > example1.lst2` in Linux

GAMS Example: Log Output (1)

- The log output to the console stored in `example1.log` is:

Listing: example1.gams, lines 1–14

```
--- Job example1.gams Start 11/28/17 10:10:55 24.9.2 r64480 LEX-LEG x86 64bit/Linux
GAMS 24.9.2   Copyright (C) 1987-2017 GAMS Development. All rights reserved
Licensee: GAMS Development Corporation, USA                 G871201/0000CA-ANY
          Free Demo, +1 202-342-0180, support@gams.com, www.gams.com DC0000
--- Starting compilation
--- example1.gams(12) 2 Mb
--- Starting execution: elapsed 0:00:00.001
--- Generating LP model example1
--- example1.gams(12) 3 Mb
---   4 rows  3 columns  7 non-zeroes
--- Executing CPLEX: elapsed 0:00:00.002

IBM ILOG CPLEX   24.9.2 r64480 Released Nov 14, 2017 LEG x86 64bit/Linux
Cplex 12.7.1.0
```

GAMS Example: Log Output (2)

- The log output to the console stored in `example1.log` is:

Listing: `example1.gams`, lines 16–32

```
Reading data...
Starting Cplex...
Space for names approximately 0.00 Mb
Use option 'names no' to turn use of names off
CPXPARAM_Advance                                 0
CPXPARAM_Simplex_Display                         2
CPXPARAM_Simplex_Limits_Iterations               2000000000
CPXPARAM_TimeLimit                               1000
CPXPARAM_Threads                                 1
CPXPARAM_Parallel                                1
CPXPARAM_Tune_TimeLimit                          200
Tried aggregator 1 time.
LP Presolve eliminated 4 rows and 3 columns.
All rows and columns eliminated.
Presolve time = 0.00 sec. (0.00 ticks)
LP status(1): optimal
Cplex Time: 0.00sec (det. 0.00 ticks)
```

GAMS Example: Log Output (3)

- The log output to the console stored in `example1.log` is:

Listing: example1.gams, lines 34–41

```
Optimal solution found.
Objective :           5.200000

--- Restarting execution
--- example1.gams(12) 2 Mb
--- Reading solution for model example1
*** Status: Normal completion
--- Job example1.gams Stop 11/28/17 10:10:56 elapsed 0:00:00.020
```

GAMS Example: Solution Output (1)

- The output of the result stored in `example1.lst2` is:

Listing: example1.gams, lines 1–20

```
GAMS 24.9.2  r64480 Released Nov 14, 2017 LEX-LEG x86 64bit/Linux

    11/28/17 10:10:55 Page 1
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
C o m p i l a t i o n


   1  Variables f, x1, x2;
   2
   3  Equations obj, g1, g2, g3;
   4
   5  obj.. f =e= 10 - 0.6*x1 - 0.4*x2;
   6  g1..  x2 =g= 0;
   7  g2..  x1 =g= 0;
   8  g3..  1.25*x1 + 2*x2 =l= 10;
   9
  10  Model example1 / obj, g1, g2, g3 /;
  11
  12  solve example1 using LP minimizing f;


COMPILATION TIME     =        0.000 SECONDS      2 MB  24.9.2 r64480 LEX-LEG
```

GAMS Example: Solution Output (2)

- The output of the result stored in `example1.lst2` is:

Listing: `example1.gams`, lines 21–43

```
GAMS 24.9.2  r64480 Released Nov 14, 2017 LEX-LEG x86 64bit/Linux

    11/28/17 10:10:55 Page 2
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
Equation Listing    SOLVE example1 Using LP From line 12


---- obj  =E=

obj..  f + 0.6*x1 + 0.4*x2 =E= 10 ; (LHS = 0, INFES = 10 ****)


---- g1  =G=

g1..  x2 =G= 0 ; (LHS = 0)


---- g2  =G=

g2..  x1 =G= 0 ; (LHS = 0)


---- g3  =L=

g3..  1.25*x1 + 2*x2 =L= 10 ; (LHS = 0)
```

GAMS Example: Solution Output (3)

- The output of the result stored in `example1.lst2` is:

Listing: example1.gams, lines 45–72

```
GAMS 24.9.2  r64480 Released Nov 14, 2017 LEX-LEG x86 64bit/Linux

    11/28/17 10:10:55 Page 3
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
Column Listing      SOLVE example1 Using LP From line 12


---- f

f
                (.LO, .L, .UP, .M = -INF, 0, +INF, 0)
        1       obj


---- x1

x1
                (.LO, .L, .UP, .M = -INF, 0, +INF, 0)
        0.6     obj
        1       g2
        1.25    g3


---- x2

x2
                (.LO, .L, .UP, .M = -INF, 0, +INF, 0)
        0.4     obj
        1       g1
        2       g3
```

GAMS Example: Solution Output (4)

- The output of the result stored in `example1.lst2` is:

Listing: `example1.gams`, lines 74–89

```
GAMS 24.9.2  r64480 Released Nov 14, 2017 LEX-LEG x86 64bit/Linux

    11/28/17 10:10:55 Page 4
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
Model Statistics    SOLVE example1 Using LP From line 12


MODEL STATISTICS

BLOCKS OF EQUATIONS           4     SINGLE EQUATIONS            4
BLOCKS OF VARIABLES           3     SINGLE VARIABLES            3
NON ZERO ELEMENTS             7


GENERATION TIME      =        0.001 SECONDS      3 MB  24.9.2 r64480 LEX-LEG


EXECUTION TIME       =        0.001 SECONDS      3 MB  24.9.2 r64480 LEX-LEG
```

GAMS Example: Solution Output (5)

- The output of the result stored in `example1.lst2` is:

Listing: example1.gams, lines 90–116

```
GAMS 24.9.2  r64480 Released Nov 14, 2017 LEX-LEG x86 64bit/Linux

    11/28/17 10:10:55 Page 5
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
Solution Report     SOLVE example1 Using LP From line 12


               S O L V E      S U M M A R Y

     MODEL   example1            OBJECTIVE  f
     TYPE    LP                  DIRECTION  MINIMIZE
     SOLVER  CPLEX               FROM LINE  12

**** SOLVER STATUS     1 Normal Completion
**** MODEL STATUS      1 Optimal
**** OBJECTIVE VALUE                5.2000

 RESOURCE USAGE, LIMIT          0.003      1000.000
 ITERATION COUNT, LIMIT         0    2000000000

IBM ILOG CPLEX   24.9.2 r64480 Released Nov 14, 2017 LEG x86 64bit/Linux
Cplex 12.7.1.0

Space for names approximately 0.00 Mb
Use option 'names no' to turn use of names off
LP status(1): optimal
Cplex Time: 0.00sec (det. 0.00 ticks)
Optimal solution found.
Objective :           5.200000
```

GAMS Example: Solution Output (6)

- The output of the result stored in `example1.lst2` is:

Listing: `example1.gams`, lines 116–138

```
                           LOWER          LEVEL          UPPER         MARGINAL

---- EQU obj                10.0000        10.0000        10.0000         1.0000
---- EQU g1                   .              .            +INF           0.5600
---- EQU g2                   .            8.0000         +INF            .
---- EQU g3                 -INF           10.0000        10.0000        -0.4800

                           LOWER          LEVEL          UPPER         MARGINAL

---- VAR f                  -INF            5.2000        +INF            .
---- VAR x1                 -INF            8.0000        +INF            .
---- VAR x2                 -INF             .            +INF            .


**** REPORT SUMMARY :        0     NONOPT
                             0 INFEASIBLE
                             0  UNBOUNDED


EXECUTION TIME       =        0.001 SECONDS      2 MB   24.9.2 r64480 LEX-LEG
```

GAMS Example: Solution Output (7)

- The output of the result stored in `example1.lst2` is:

Listing: `example1.gams`, lines 141–53

```
USER: GAMS Development Corporation, USA              G871201/0000CA-ANY
      Free Demo, +1 202-342-0180, support@gams.com, www.gams.com DC0000


**** FILE SUMMARY

Input      example1.gams
GAMS 24.9.2  r64480 Released Nov 14, 2017 LEX-LEG x86 64bit/Linux

    11/28/17 10:10:55 Page 6
G e n e r a l   A l g e b r a i c   M o d e l i n g   S y s t e m
Solution Report     SOLVE example1 Using LP From line 12


Output     example1.lst
```

Section Outline

A Second Example

- We now discuss a second example, directly taken from the GAMS tutorial by Rosenthal [1], which in turn based on a transportation problem defined by Dantzig [2]

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- We know
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  - the supplies a_i of that product produced at the factory i,
  - there are several markets j and we know
  - the demands b_j for the product at each market j, and
  - the costs $c_{i,j}$ of shipping one unit the commodity from each factory i to each market j.
- For each factory i and market j, we want to find the exact amount $x_{i,j}$ of the product to ship from i to j such that the overall transportation cost f is minimized (obviously $x_{i,j} \geq 0 \; \forall i, j$).

A Second Example: Constraints

- The goal is to minimize the overall transportation costs f

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- The goal is to minimize the overall transportation costs f, which is the unit transportation cost from each factory i to each market j times the amount $x_{i,j}$ of the product transported from i to j

$$\text{minimize } f(x) \quad = \quad \sum_{\forall i,j} c_{i,j} * x_{i,j} \tag{7}$$

A Second Example: Constraints

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A Second Example: Constraints

- The goal is to minimize the overall transportation costs f
- Clearly, if we transport nothing to nowhere, we get cost 0... ...but there are two constraints limiting our choices

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A Second Example: Constraints

- The goal is to minimize the overall transportation costs f
- There are two constraints limiting our choices:
- We cannot take more supplies from a factory i than the supply a_i available in i

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A Second Example: Constraints

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- There are two constraints limiting our choices:
- We cannot take more supplies from a factory i than the supply a_i available in i, i.e., the sum of all amounts $x_{i,j}$ taken from factory i to all the markets j cannot exceed a_i

$$\text{minimize } f(x) = \sum_{\forall i,j} c_{i,j} * x_{i,j} \tag{5}$$

$$\text{subject to } \forall i \left(\sum_{\forall j} x_{i,j}\right) \leq a_i \tag{7}$$

A Second Example: Constraints

- The goal is to minimize the overall transportation costs f
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- We must satisfy the demand b_j of each market

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$$\forall j \quad \left(\sum_{\forall i} x_{i,j}\right) \geq b_j \tag{7}$$

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$$\text{minimize } f(x) = \sum_{\forall i,j} c_{i,j} * x_{i,j} \tag{5}$$

$$\text{subject to } \forall i \quad \left(\sum_{\forall j} x_{i,j}\right) \leq a_i \tag{6}$$

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- Let us assume that we have the two plants `nanjingPlant` and `suzhouPlant` with supplies of 350 and 600 units of the product, respectively.

Problem Instance

- The above definition is a general problem, but for each planning task there will be a concrete scenario.
- Let us assume that we have the two plants `nanjingPlant` and `suzhouPlant` with supplies of 350 and 600 units of the product, respectively.
- There also be three markets, `hefeiSuguo`, `beijingSuning`, and `shanghaiTaobao` with demands of 325, 300, and 275, respectively.

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- Let us assume that we have the two plants `nanjingPlant` and `suzhouPlant` with supplies of 350 and 600 units of the product, respectively.
- There also be three markets, `hefeiSuguo`, `beijingSuning`, and `shanghaiTaobao` with demands of 325, 300, and 275, respectively.
- The distances in KM between the plants and markets be as follows:

	`hefeiSuguo`	`beijingSuning`	`shanghaiTaobao`
`nanjingPlant`	169.7	1042.3	303.1
`suzhouPlant`	384.4	1150.9	95.6

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- Transporting one unit of the product for one kilometer costs 2 RMB.

GAMS Model

Listing: transport.gams

```
Sets        i       factories    / nanjingPlant, suzhouPlant /
            j       markets      / hefeiSuguo, beijingSuning, shanghaiTaobao / ;

Parameters  a(i)    supply units available at factory i   /  nanjingPlant   350
                                                              suzhouPlant    600  /

            b(j)    demand at market j in units           /  hefeiSuguo      325
                                                              beijingSuning   300
                                                              shanghaiTaobao  275  / ;

Table       d(i,j)  distance in kilometers
                                        hefeiSuguo    beijingSuning    shanghaiTaobao
                          nanjingPlant      169.7          1042.3           303.1
                          suzhouPlant       384.4          1150.9           95.6   ;

Scalar      tc      transport costs in RMB per unit per kilometer /2/ ;

Parameter   c(i,j)  transport cost in RMB per unit;    c(i,j) = tc * d(i,j);

Variables   x(i,j)  shipment quantities in units
            f       total transportation costs ;

Positive Variable x ;

Equations cost, supply(i), demand(j) ;

cost ..         f  =e=  sum((i,j), c(i,j)*x(i,j)) ;
supply(i) ..    sum(j, x(i,j))  =l=  a(i) ;
demand(j) ..    sum(i, x(i,j))  =g=  b(j) ;

Model transport /all/ ;
Solve transport using lp minimizing f ;
Display x.l, x.m ;
```

Solution

- We can execute and solve the model via

```
gams transport.gams -o transport.lst -logOption 2
```

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- In `transport.lst`, we find the following solution:

Listing: `transport.lst`, lines 179–183, 196–199

```
                            LOWER          LEVEL          UPPER         MARGINAL

---- VAR f                  -INF       847995.0000        +INF             .

  f  total transportation costs
             hefeiSuguo  beijingSu~  shanghaiT~

nanjingPlant     325.000      25.000
suzhouPlant                  275.000     275.000
```

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```

- The complete demand of 325 of `hefeiSuguo` is covered by the `nanjingPlant`

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suzhouPlant                  275.000     275.000
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- The complete demand of 325 of `hefeiSuguo` is covered by the `nanjingPlant`
- The remaining 25 units from the `nanjingPlant` go to the `beijingSuning`

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- The complete demand of 325 of `hefeiSuguo` is covered by the `nanjingPlant`
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- The remaining 25 units from the `nanjingPlant` go to the `beijingSuning`
- The remaining 275 units required by `beijingSuning` need to be provided by the `suzhouPlant`
- The `suzhouPlant` also supplies `shanghaiTaobao` with its required 275 units
- All in all, this incurs a transportation cost of 847'995 RMB

Section Outline

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- Metaheuristics can easily be developed to address a much wider range of problems, although tools like GAMS also support quadratic and nonlinear-programming methods
- Sometimes, constraints or objectives which are *not* linear are represented as linear functions so that Linear Programming can be used. The optimal solutions for such mis-represented problems are not necessarily optimal solutions of the actual optimization problem...
- It is important to choose the right tool for the right task.

Metaheuristics vs. Linear Programming

Method	Advantages	Disadvantages
Linear Programming		
Metaheuristics		

Metaheuristics vs. Linear Programming

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Linear Programming	• industry standard: well-established technology that everyone in operations research knows • will provide the optimal solution for the specified problem • fast, mature • widely understood and used quasi-standard software such as CPLEX • ideal for problems where objective and constraints are linear • nice modelling languages and environments such as GAMS	
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Metaheuristics	• applicable to virtually arbitrary problem definitions • prototypes can be developed quickly from problem definition • best approach for many classical problems (Max-SAT, TSP, ...) • can deal with multiple objectives • can deal with uncertainties and dynamic problems • can deal continuous/numerical problem with real-valued decision variables over convex objective functions • can solve problems efficiently where Linear Programming does not work well • we can quickly built prototypes which deliver acceptable solution quality • constraints are often natural part of the problem representation (e.g., for TSPs)	

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谢谢

Thank you

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Caspar David Friedrich, “Der Wanderer über dem Nebelmeer”, 1818
http://en.wikipedia.org/wiki/Wanderer_above_the_Sea_of_Fog

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