Metaheuristic Optimization

15. Multi-Objective Optimization

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Outline

website

Section Outline

Definition (Objective Function)

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- Usually subject to minimization
- Not necessary a function as we know it from high school (like $f(x) = x^2 + \dots$) but may be arbitrarily complex...

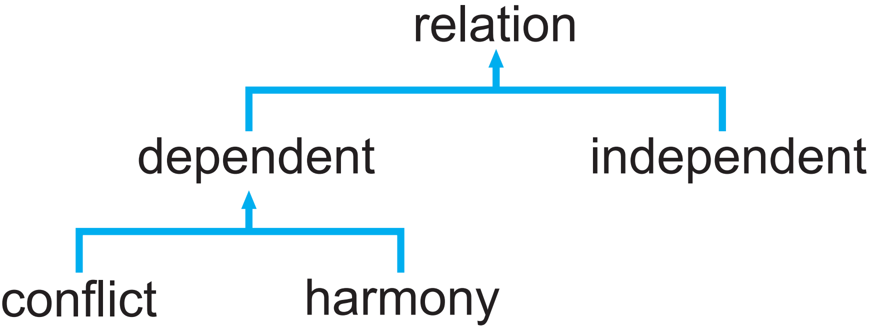
Definition (Multi-Objective Optimization Problem)

In a multi-objective optimization problem (MOP), a set $\vec{f} : \mathbb{X} \mapsto \mathbb{R}^n$ consisting of n objective functions $f_i : \mathbb{X} \mapsto \mathbb{R}$ is to be optimized over a solution space $\mathbb{X}$ [1–3].

$$\vec{f} = \{f_i : \mathbb{X} \mapsto \mathbb{R} : i \in 1 \dots n\} \quad (1)$$

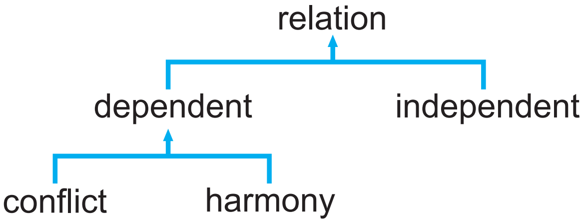
$$\vec{f}(x) = (f_1(x), f_2(x), \dots)^T \implies \vec{f} : \mathbb{X} \mapsto \mathbb{R}^n \quad (2)$$

- These objective functions can have different relations with each other. [4, 5]



Independent Objectives

- Independent objective functions are unrelated to each other.



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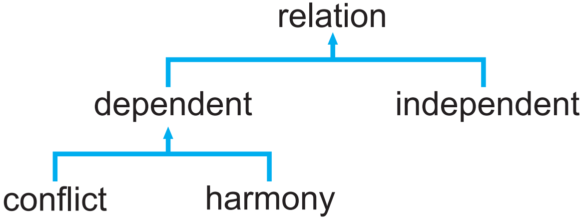

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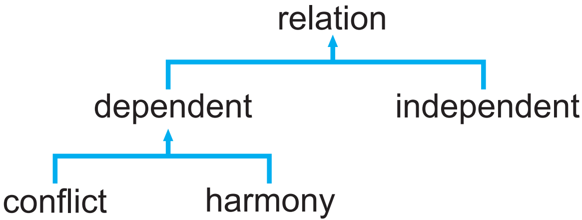
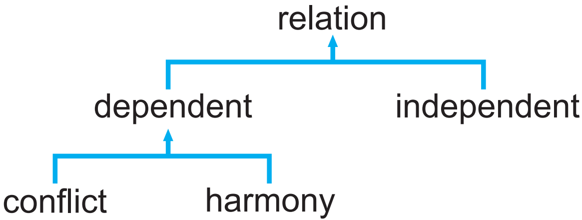
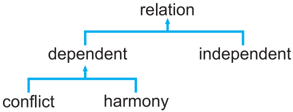


- Independent objective functions are unrelated to each other.
- Example: Find a (1) fast car with (2) beautiful color. $\longrightarrow$ color and speed may be optimized separately
- Uninteresting. Problem can be decomposed into sub-problems which can be optimized separately and solutions of sub-problems can be composed to solution of overall problem



Harmonizing Objectives

- If two functions harmonize, then achieving an improvement in one objective will also lead to an improvement in the other.



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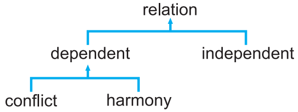

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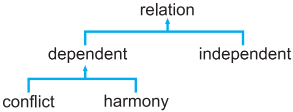

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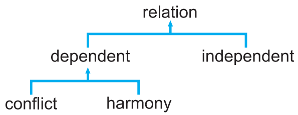

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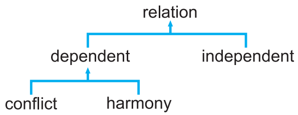
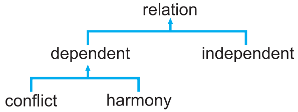
- Uninteresting. One of the objectives can be omitted / left away, as its presence does neither change the result nor does it make the problem easier



Conflicting Objectives

- If two objectives conflict, then achieving an improvement in one of means getting worse in the other one.



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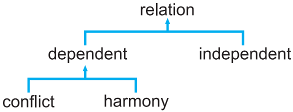

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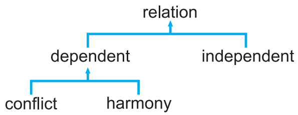

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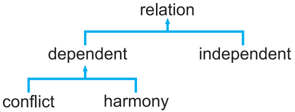
- This is the really interesting situation – here we need to do something!



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- Instead, they may harmonize in some parts of the solution space and conflict in others

Optimal?

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What does "optimal" mean in the presence of multiple optimization criteria?

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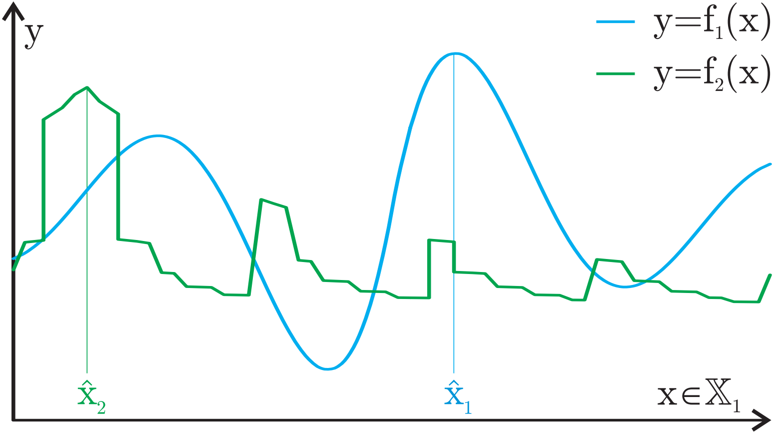
  4. and so on...
- Separate a multi-objective optimization problem into n single-objective ones

- Two 1-dimensional functions subject to maximization:
$\vec{f} = \{f_1, f_2\}, f_i : \mathbb{R} \mapsto \mathbb{R}\, \forall i \in \{1, 2\}$



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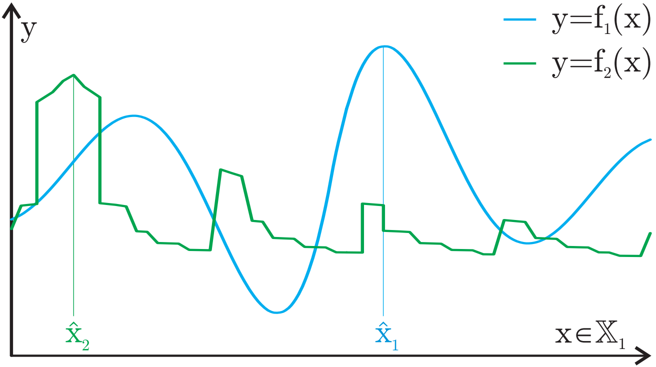

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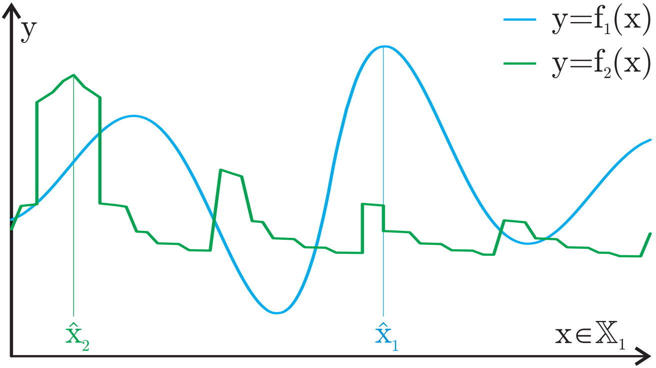
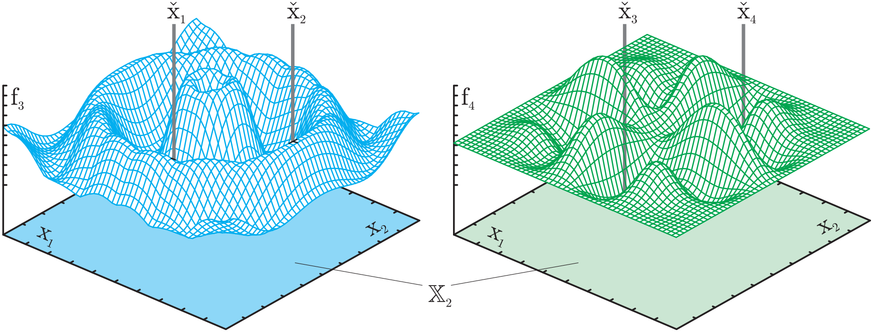


Example B: Two 2d-Functions

- Two 2-dimensional functions to minimization:
  $\vec{f} = \{f_3, f_4\}, f_i : \mathbb{R}^2 \mapsto \mathbb{R}\,\forall i \in \{3, 4\}$



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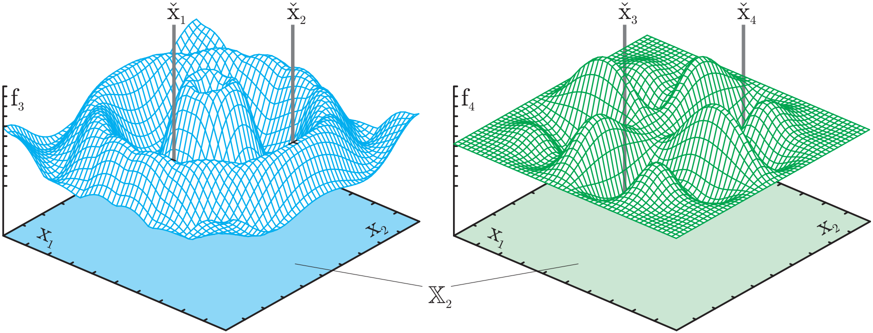

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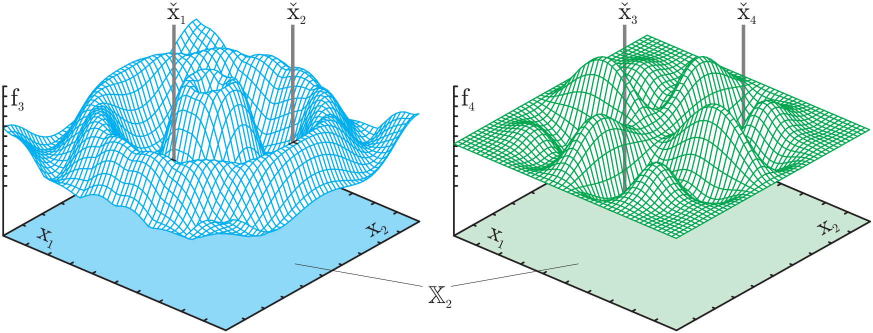

Lexicographic Optimization

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- We can use any of the single-objective techniques we already know to solve it...

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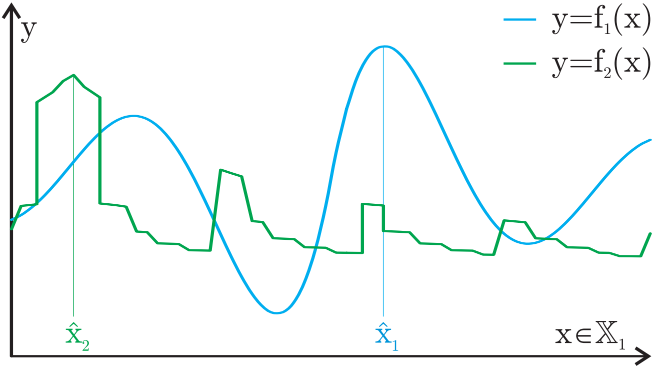

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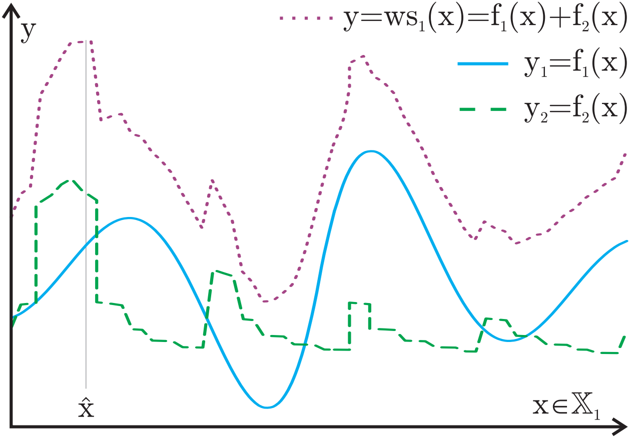

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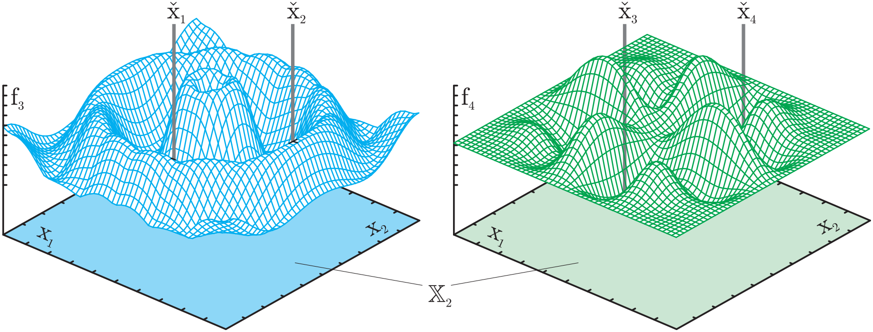

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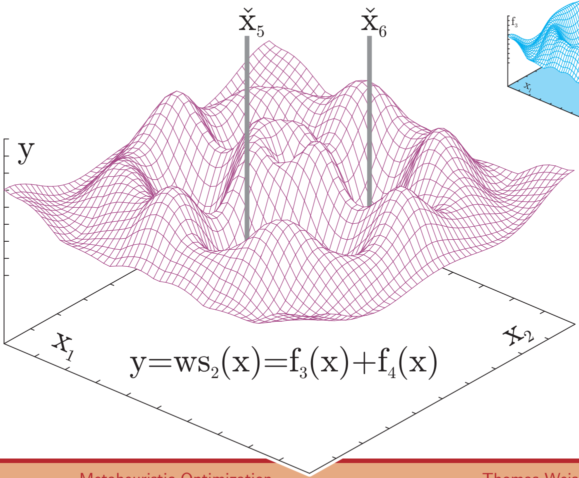
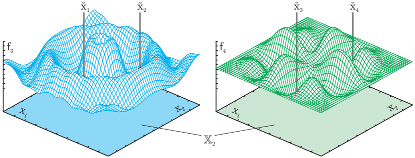
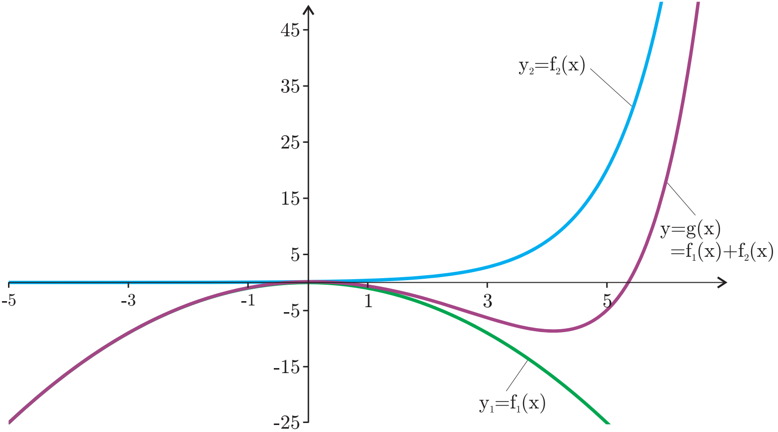




- Section of two functions subject to either maximization or minimization



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- ✘ May not be able to discover the whole trade-off curve
- ✘ Objective functions are not always precise measures of utility, adding them up thus does not always make sense

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- "Find a fast car which is environmentally friendly!"

Pareto-based Approach

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  - Assume the following possible candidate solutions:
    Car A: 200 km/h with 4 L/100km
    Car B: 150 km/h with 9 L/100km
    Car C: 250 km/h with 7 L/100km
    Car D: 175 km/h with 7 L/100km
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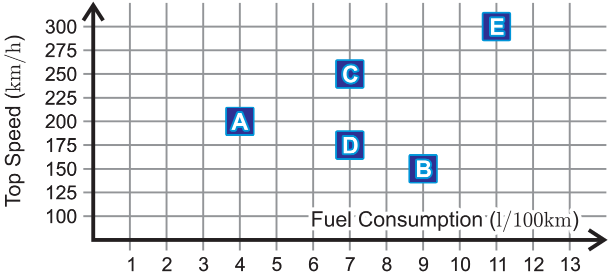

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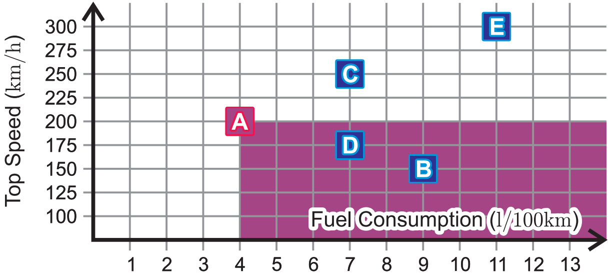

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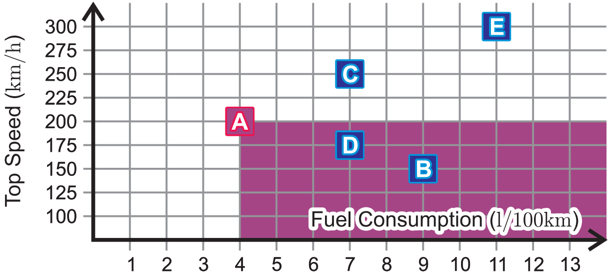
  - Car A is also better than car B: it is faster and needs less fuel



- “Find a fast car which is environmentally friendly!”
  - Assume the following possible candidate solutions:
    Car A: 200 km/h with 4 L/100km   Car B: 150 km/h with 9 L/100km
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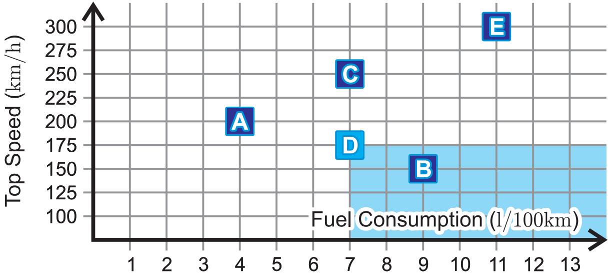
    Car E: 300 km/h with 11 L/100km
  - Also, car D is better than car B: it is faster and needs less fuel



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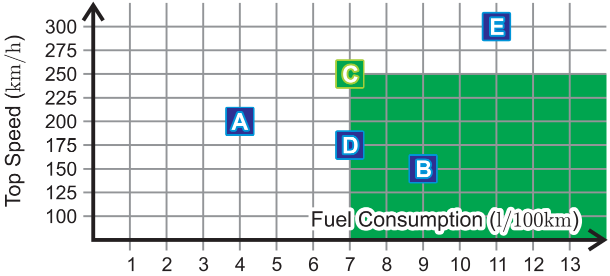

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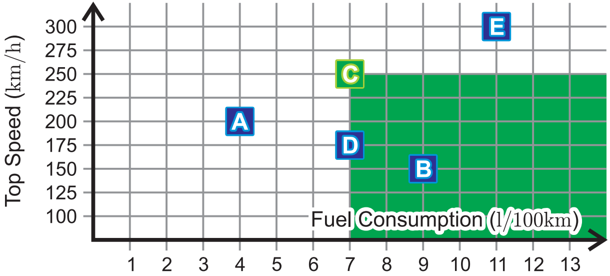
  - Car C is better than car D: it is faster at the same fuel consumption
  - Car C is also better than car B: it is faster and needs less fuel



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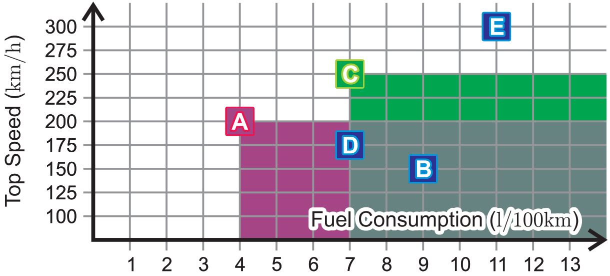

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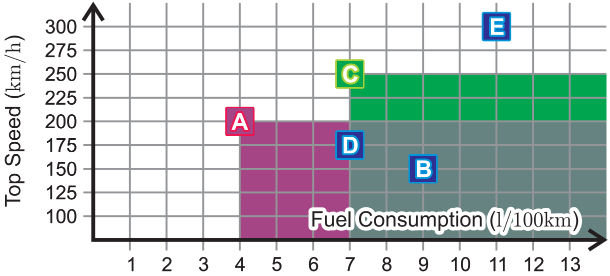

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  - However, car C and A cannot be compared!
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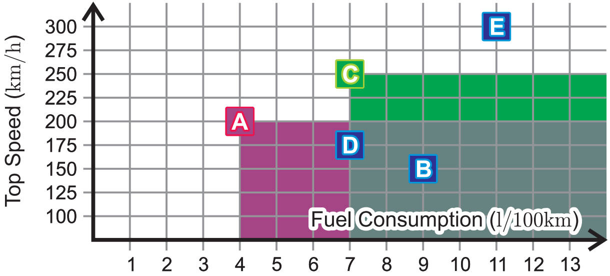
  - Car C is faster than car A, but needs more fuel



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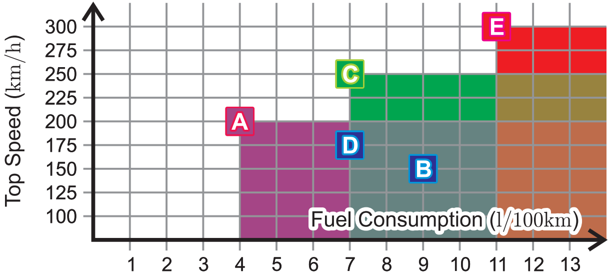
  - No car is better than car E



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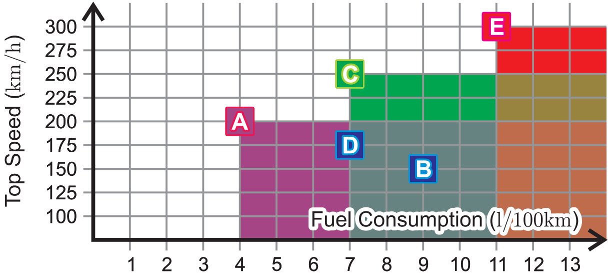
  - No car is better than car E
  - But: No car is worse than car E!



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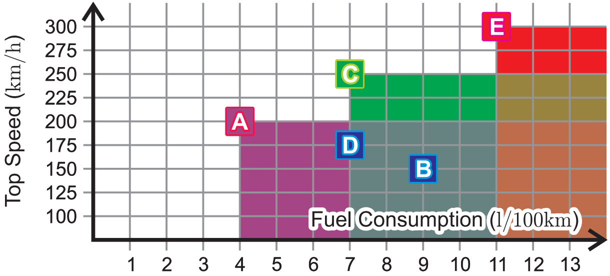
  - No car is better than car E
  - But: No car is worse than car E!
  - All cars are slower than car E, but all need less fuel



Pareto-based Approach

- Idea first developed by Edgeworth [18] and Pareto [19] in the last two decades of the 19^{th} century

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- Pareto optimality defines the frontier of solutions that can be reached by trading-off conflicting objectives in an optimal manner [3, 20–25]

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- Pareto optimality defines the frontier of solutions that can be reached by trading-off conflicting objectives in an optimal manner [3, 20–25]
- Pareto optimality became an important notion in economics, game theory, engineering, and social sciences [26–29].

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Definition (Domination)

An element x_1 dominates (is preferred to) an element x_2 ($x_1 \dashv x_2$) if x_1 is better than x_2 in at least one objective function and not worse with respect to all other objectives. Based on the set $\vec{f}$ of objective functions f, we can write:

$$x_1 \dashv x_2 \quad \Leftrightarrow \quad \forall i \in 1 \dots n \Rightarrow \omega_i f_i(x_1) \leq \omega_i f_i(x_2) \;\wedge \quad \exists j \in 1 \dots n : \omega_j f_j(x_1) < \omega_j f_j(x_2) \tag{7}$$

$$\omega_i = \begin{cases} 1 & \text{if } f_i \text{ should be minimized} \\ -1 & \text{if } f_i \text{ should be maximized} \end{cases} \tag{8}$$

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- Pareto optimality defines the frontier of solutions that can be reached by trading-off conflicting objectives in an optimal manner [3, 20–25]

Definition (Pareto Optimal)

An element $x^\star \in \mathbb{X}$ is Pareto optimal (and hence, part of the optimal set $X^\star$) if it is not dominated by any other element in the solution space $\mathbb{X}$. $X^\star$ is called the Pareto-optimal set or Pareto set.

$$x^\star \in X^\star \Leftrightarrow \nexists x \in \mathbb{X} : x \dashv x^\star \tag{7}$$

Pareto-based Approach

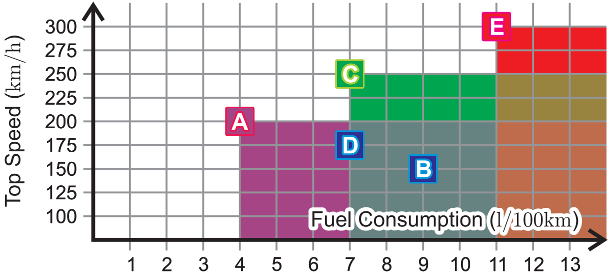
- Idea first developed in the last two decades of the 19^{th} century[18, 19]
- Pareto optimality defines the frontier of solutions that can be reached by trading-off conflicting objectives in an optimal manner [3, 20–25]

Definition (Pareto Frontier)

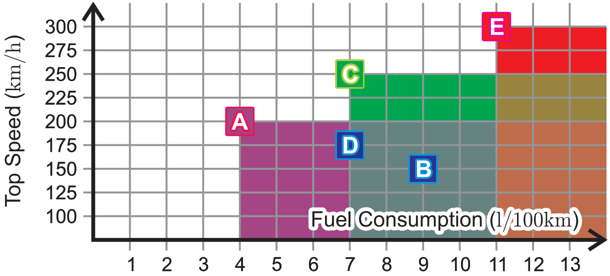
For a given optimization problem, the Pareto front(ier) $F^{\star} \subset \mathbb{R}^n$ is defined as the set of results the objective function vector $\vec{f}$ creates when it is applied to all the elements of the Pareto-optimal set $X^{\star}$.

$$F^{\star} = \{\vec{f}(x^{\star}\} : x^{\star} \in X^{\star}) \tag{7}$$

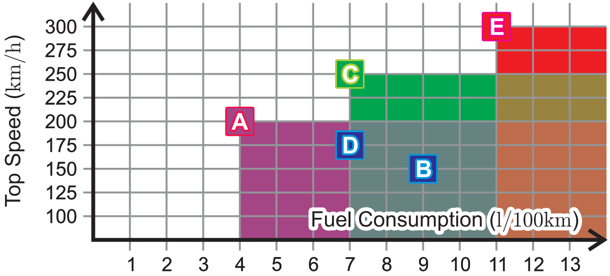
- Non-dominated



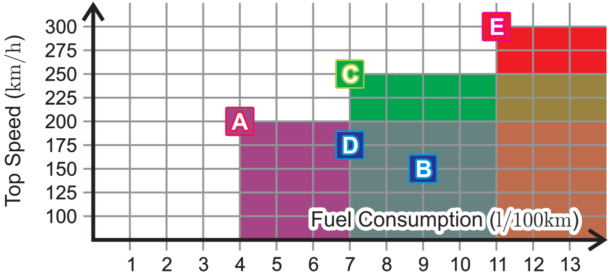
- Non-dominated: A



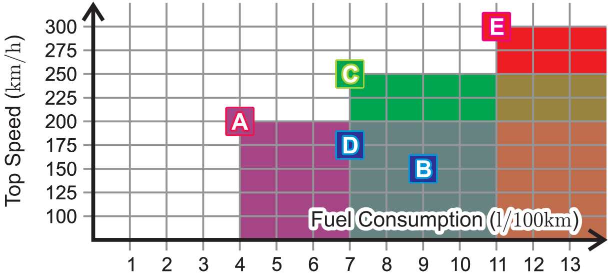
- Non-dominated: A, C



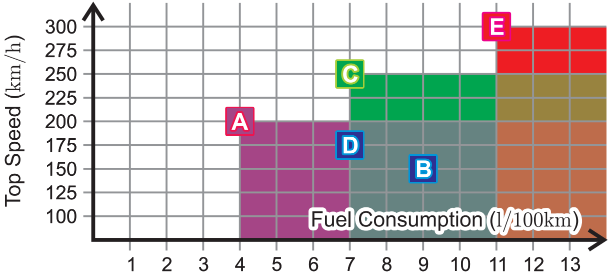
- Non-dominated: A, C, E



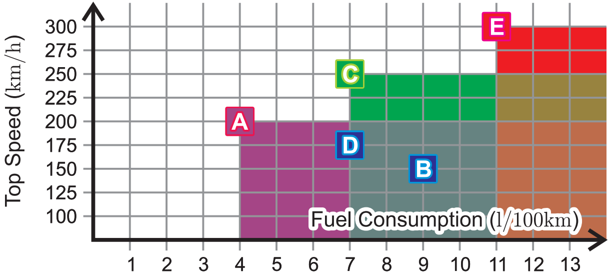
- Non-dominated: A, C, E
- Dominated



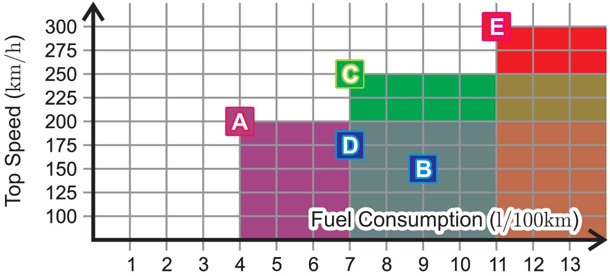
- Non-dominated: A, C, E
- Dominated: D



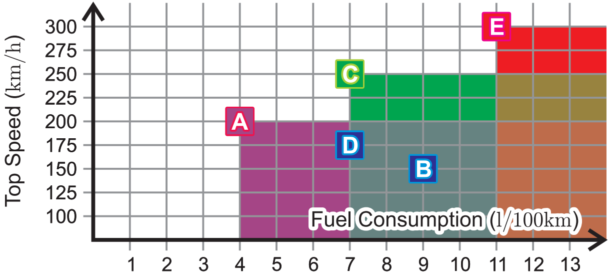
- Non-dominated: A, C, E
- Dominated: D, B



- Non-dominated: A, C, E
- Dominated: D, B
- Pareto set

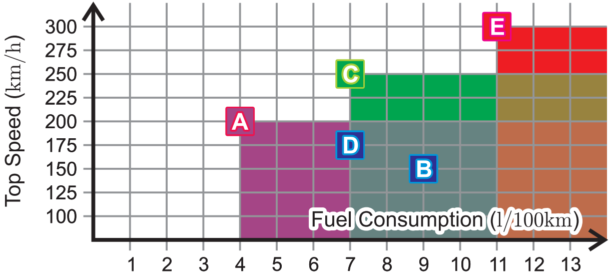


- Non-dominated: A, C, E
- Dominated: D, B
- Pareto set: $X^{\star} = \{A, C, E\}$

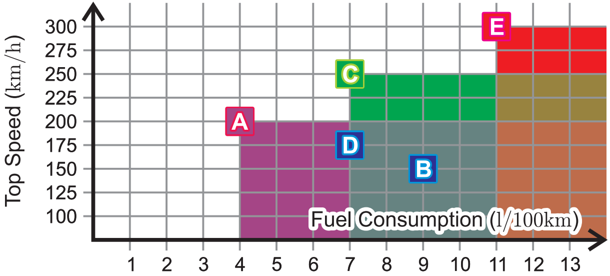


Pareto-based Approach

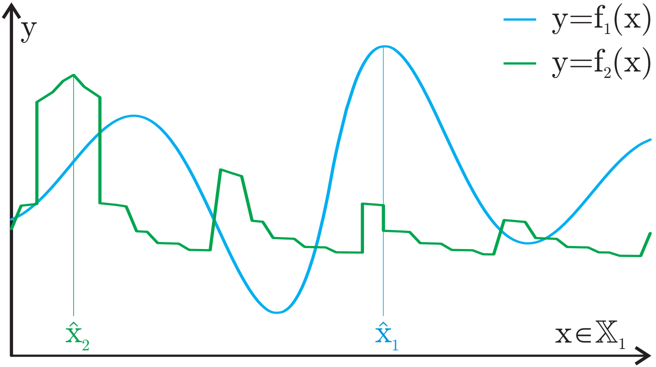
- Non-dominated: A, C, E
- Dominated: D, B
- Pareto set: $X^{\star} = \{A, C, E\}$
- Pareto front



- Non-dominated: A, C, E
- Dominated: D, B
- Pareto set: $X^{\star} = \{A, C, E\}$
- Pareto front: $F^{\star} = \{(4, 200), (7, 250), (11, 300)\}$

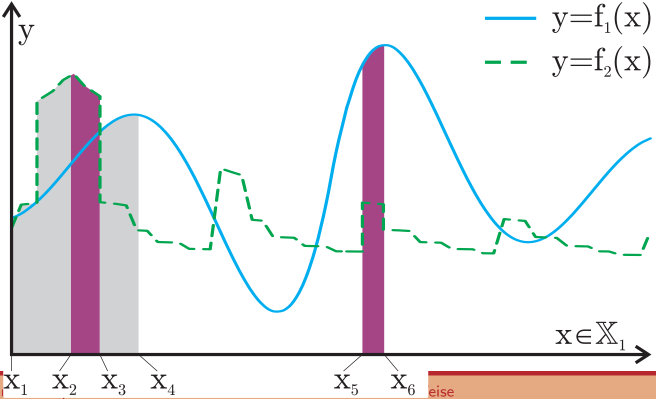


- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_{2\}}, f_i : \mathbb{R} \mapsto \mathbb{R} \, \forall i \in \{1, 2\}$

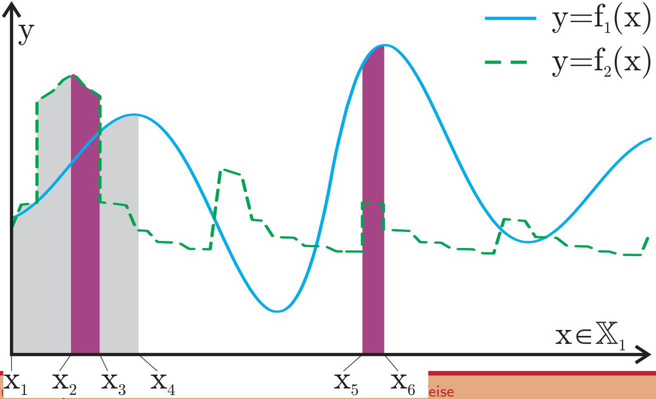


Example A: Two 1d-Functions

- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_{2\}}, f_i : \mathbb{R} \mapsto \mathbb{R}\, \forall i \in \{1, 2\}$
- $X^{\star} = [x_2, x_3] \cup [x_5, x_6]$

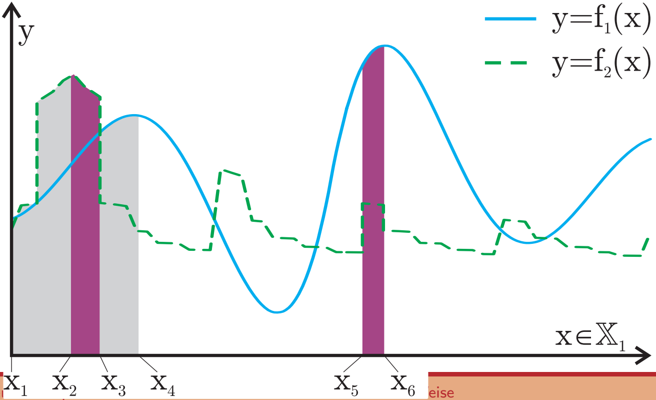


- Two 1-dimensional functions subject to maximization: $\vec{f} = \{f_1, f_{2\}}, f_i : \mathbb{R} \mapsto \mathbb{R} \, \forall i \in \{1, 2\}$
- all $x \in [x_1, x_2)$ are dominated by other points in the same region or in $[x_2, x_3]$ – f_1 and f_2 can be improved by increasing x



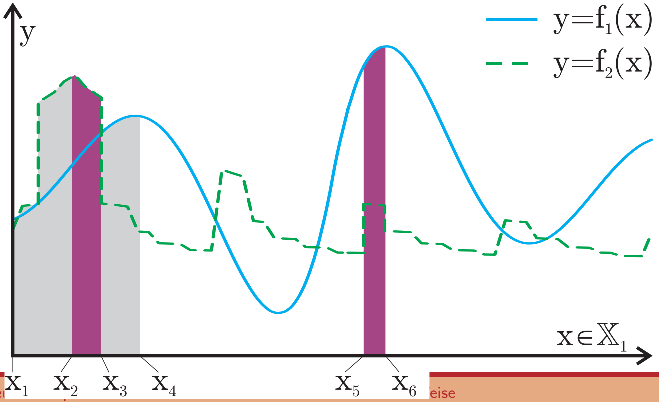
Example A: Two 1d-Functions

- Two 1-dimensional functions subject to maximization: $\vec{f} = \{f_1, f_2\}, f_i : \mathbb{R} \mapsto \mathbb{R} \, \forall i \in \{1, 2\}$
- f_1 and f_2 *harmonize* in $[x_1, x_2)$: $f_1 \sim_{[x_1,x_2)} f_2$



Example A: Two 1d-Functions

- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_2\}, f_i : \mathbb{R} \mapsto \mathbb{R}\, \forall i \in \{1, 2\}$
- $f_1(x_1 + \Delta) > f_1(x_1)$ and $f_2(x_1 + \Delta) > f_2(x_1)$ for all $\Delta \leq x_2 - x_1$



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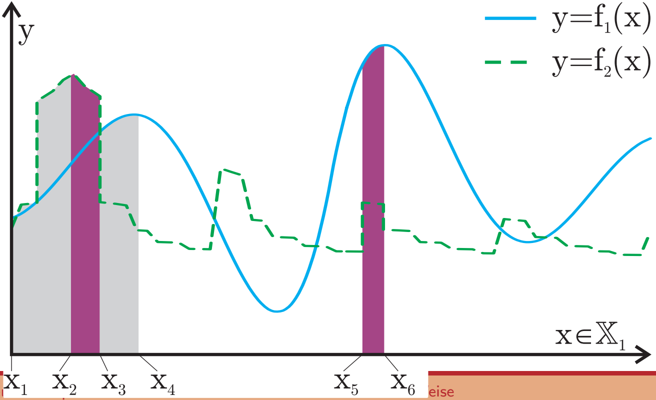
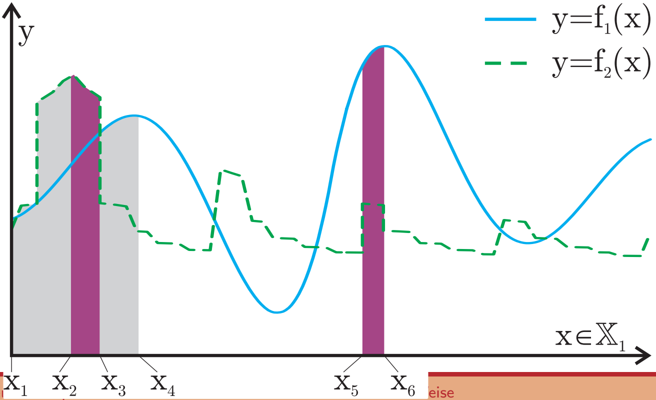
- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_{2\}}, f_i : \mathbb{R} \mapsto \mathbb{R}\, \forall i \in \{1, 2\}$
- If we reach x_2, the situation changes



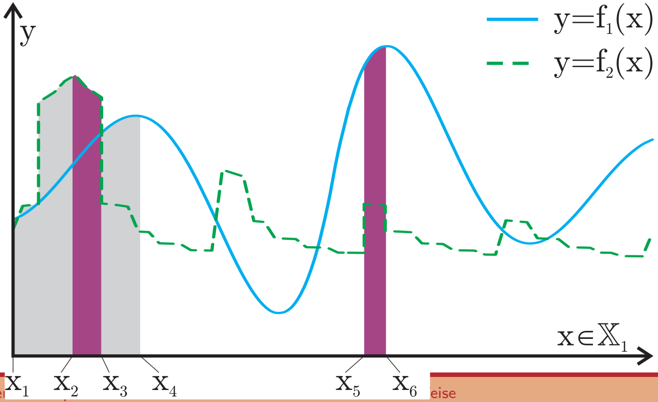
Example A: Two 1d-Functions

- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_2\}, f_i : \mathbb{R} \mapsto \mathbb{R}\, \forall i \in \{1, 2\}$
- x_2 demarks the global maximum of f_2 – the point with the highest possible f_2 value – which can never be dominated

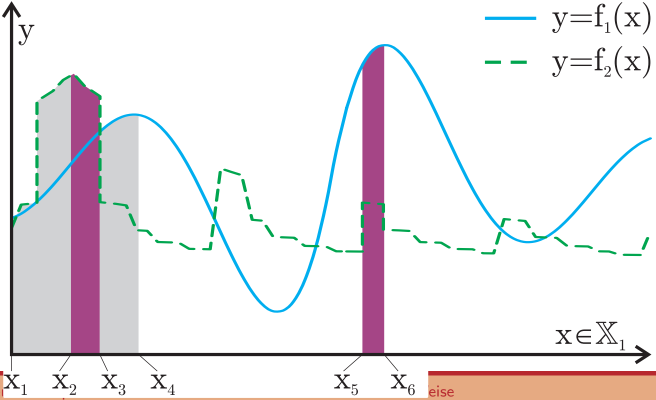


Example A: Two 1d-Functions

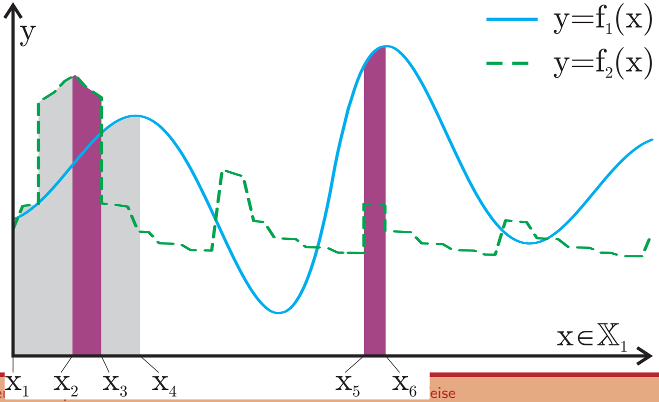
- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_{2\}}, f_i : \mathbb{R} \mapsto \mathbb{R}\,\forall i \in \{1, 2\}$
- From here on, f_2 will decrease for some time, but f_1 keeps rising.



- Two 1-dimensional functions subject to maximization: $\vec{f} = \{f_1, f_2\}, f_i : \mathbb{R} \mapsto \mathbb{R} \, \forall i \in \{1, 2\}$
- If we now go a small step Δ to the right, we will find a point $x_2 + \Delta$ with $f_2(x_2 + \Delta) < f_2(x_2)$ but also $f_1(x_2 + \Delta) > f_1(x_2)$.

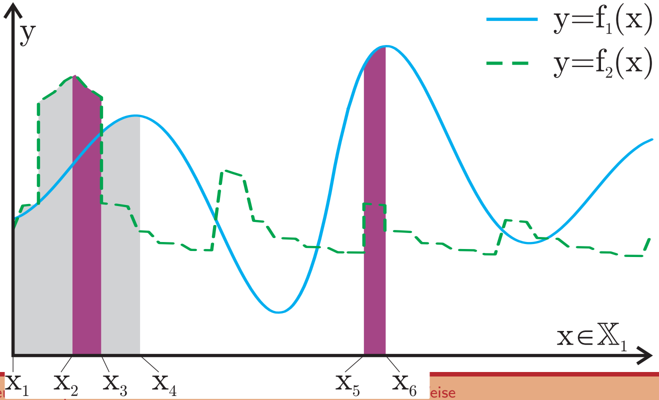


- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_2\}, f_i : \mathbb{R} \mapsto \mathbb{R}\, \forall i \in \{1, 2\}$
- One objective can only get better if another one degenerates, i.e., f_1 and f_2 *conflict* $f_1 \nsim_{[x_2, x_3]} f_2$.



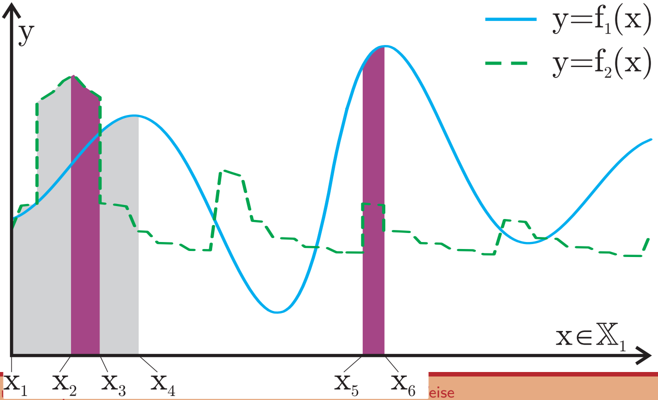
Example A: Two 1d-Functions

- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_{2\}}, f_i : \mathbb{R} \mapsto \mathbb{R}\,\forall i \in \{1, 2\}$
- No point in $[x_1, x_2)$ dominates any point in $[x_2, x_4]$



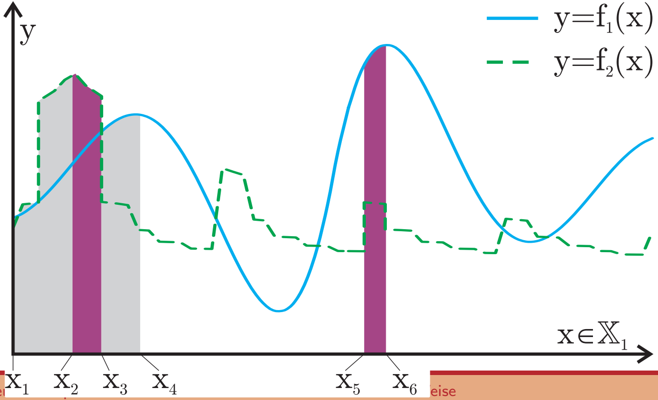
Example A: Two 1d-Functions

- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_{2\}}, f_i : \mathbb{R} \mapsto \mathbb{R} \, \forall i \in \{1, 2\}$
- f_1 keeps rising until x_4 is reached.



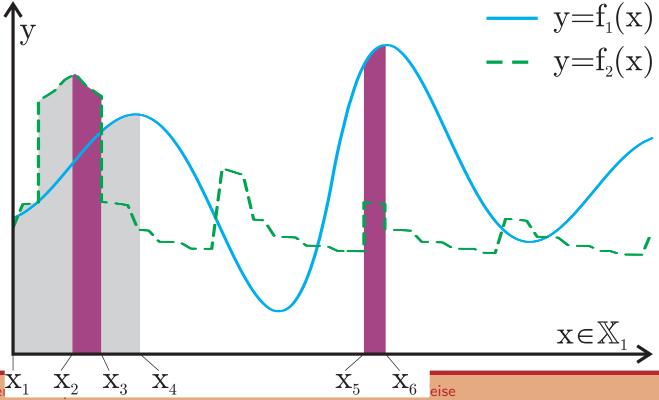
Example A: Two 1d-Functions

- Two 1-dimensional functions subject to maximization: $\vec{f} = \{f_1, f_2\}, f_i : \mathbb{R} \mapsto \mathbb{R}\, \forall i \in \{1, 2\}$
- At x_3 however, f_2 steeply falls to a very low level – lower than $f_2(x_5)$.



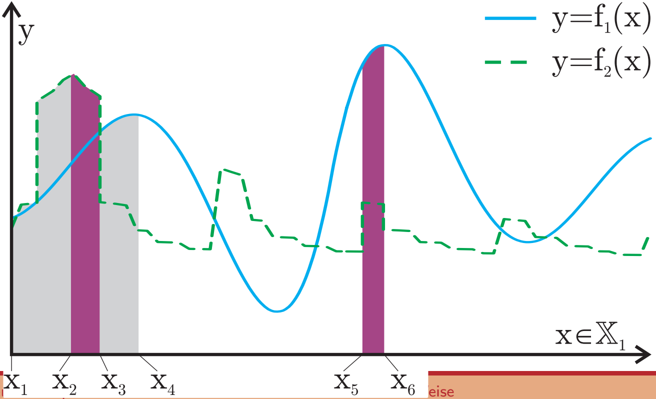
Example A: Two 1d-Functions

- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_{2\}}, f_i : \mathbb{R} \mapsto \mathbb{R}\, \forall i \in \{1, 2\}$
- The f_1 values of the points in $[x_5, x_6]$ are also higher than those of the points in $(x_3, x_4]$

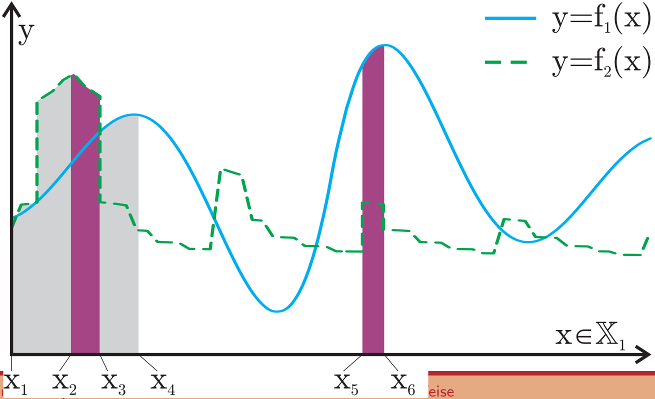


Example A: Two 1d-Functions

- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_{2\}}, f_i : \mathbb{R} \mapsto \mathbb{R}\, \forall i \in \{1, 2\}$
- All points in the set $[x_5, x_6]$ (which also contains the global maximum of f_1) dominate those in $(x_3, x_4]$.

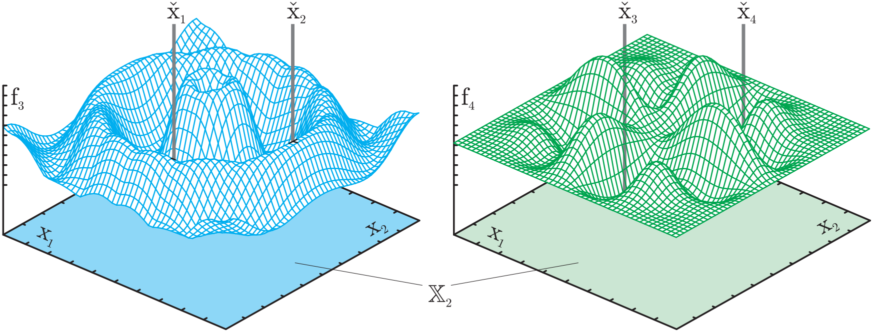


- Two 1-dimensional functions subject to maximization:
  $\vec{f} = \{f_1, f_2\}, f_i : \mathbb{R} \mapsto \mathbb{R}\, \forall i \in \{1,2\}$
- All points in $[x_4, x_5]$ and after x_6 are also dominated by the non-dominated regions just discussed.



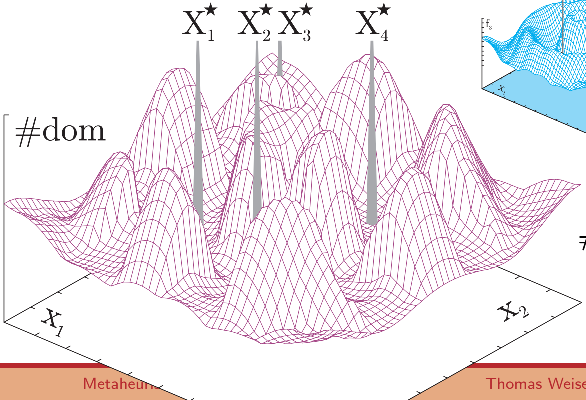
Example B: Two 2d-Functions

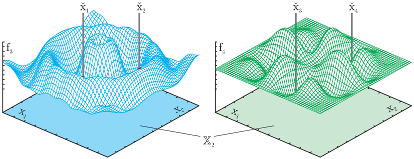
- Grid-based resolution of two 2-dimensional functions to minimization: $\vec{f} = \{f_3, f_4\}, f_i : \mathbb{R}^2 \mapsto \mathbb{R}\, \forall i \in \{3, 4\}$



Example B: Two 2d-Functions

- Grid-based resolution of two 2-dimensional functions to minimization: $\vec{f} = \{f_3, f_4\}, f_i : \mathbb{R}^2 \mapsto \mathbb{R}\,\forall i \in \{3,4\}$
- $X^{\star} = X_1^{\star} \cup X_2^{\star} \cup X_3^{\star} \cup X_4^{\star}$ are not dominated by any other candidate solution

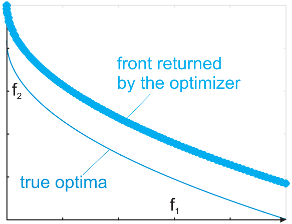




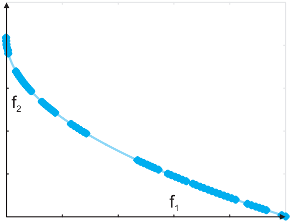
$$\mathbf{\#dom}(\mathbf{x}, \mathbf{X}) = |\{\mathbf{x}' : (x' \in X) \wedge (x' \dashv x)\}|$$

Pareto-based Optimization Results

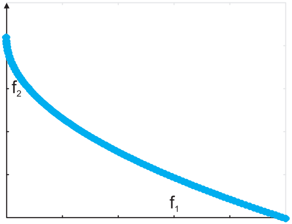
- Goal: Uniformity of convergence – many solutions close to Pareto Front that cover many different "optimal" characteristics



Bad convergence, good spread (uniformity) [30]



Good convergence, bad spread (non-uniformity) [30]



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- ✔ Results are best trade-off solutions $\longrightarrow$ Operator can make informed decision
- ✔ Multiple solutions can be discovered
- ✘ Maybe too many solutions will be discovered
- ✘ In many problems, the number of Pareto-optimal solutions may be infinite $\longrightarrow$ Which to chose?

Section Outline

- Multi-objective Evolutionary Algorithms (MOEAs) can deal with multiple, conflicting objectives [1, 2, 7, 31–53]

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MOEAs and Fitness Assignment

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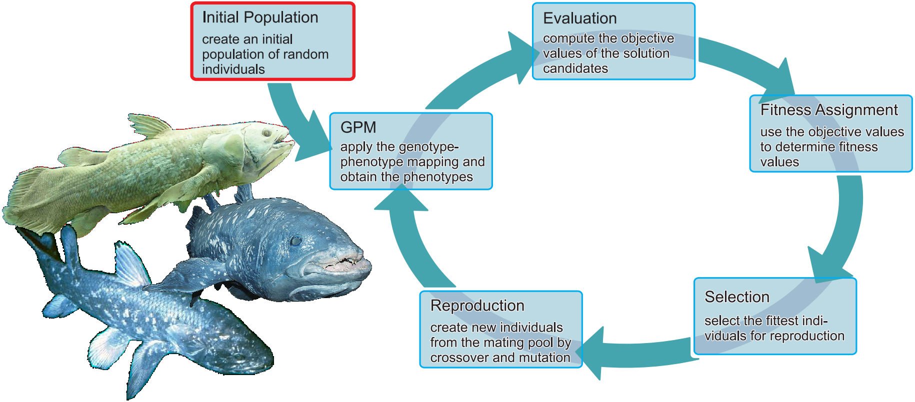
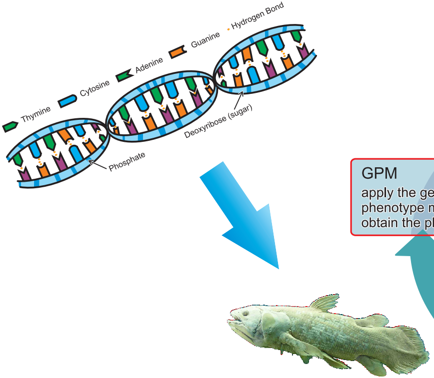
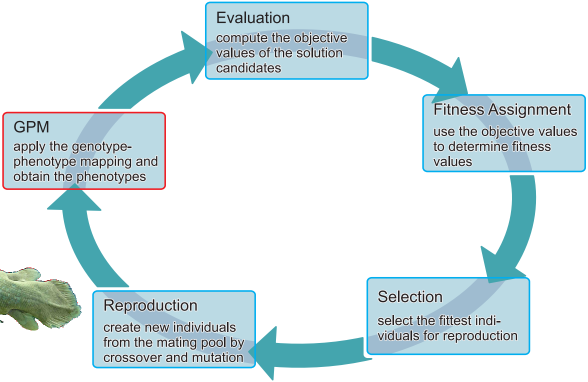
- Introduce fitness assignment process into the EA which maps the objective value vectors $\vec{f}(p.x)$ to scalar fitness values $\nu(p)$
- After such a scalar fitness has been assigned, the traditional selection schemes (fitness proportionate, tournament, ...) can be used!

- Nullary search operation to create initial individuals: create a population of random bit strings



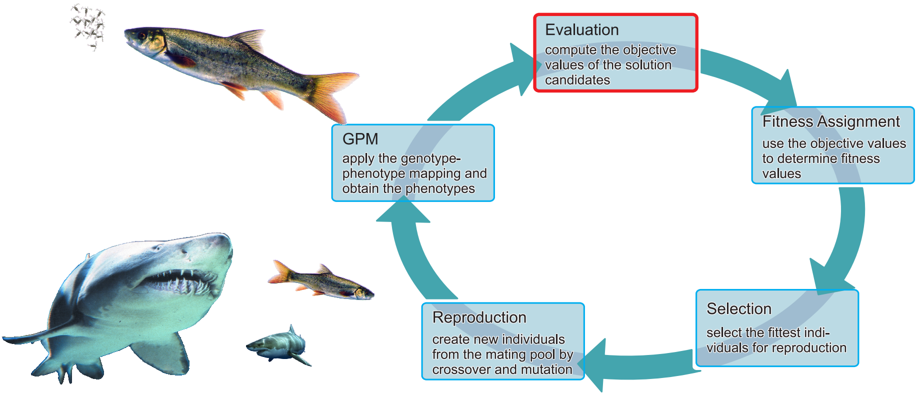
- Map the genotypes to phenotypes
- The GPM is usually problem-dependent



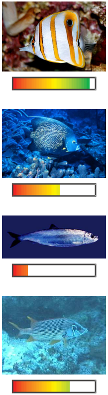


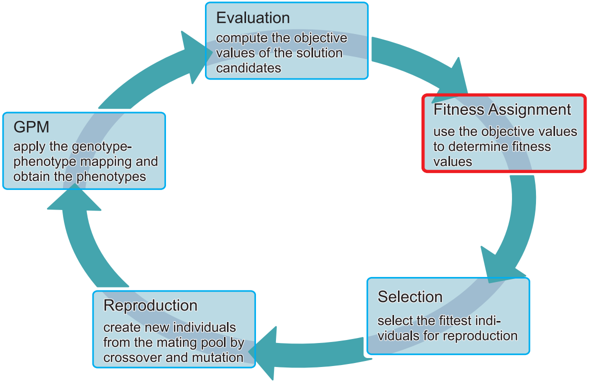
Evolutionary Algorithms: Evaluation

- Evaluate the objective functions – each solution may have different featues
- Each candidate solution x now has a vector of objective values $\vec{f}(x)$

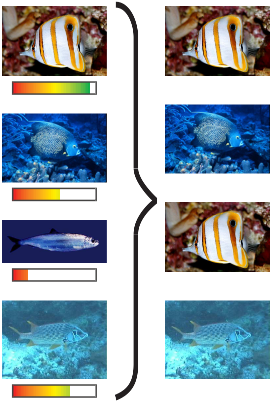


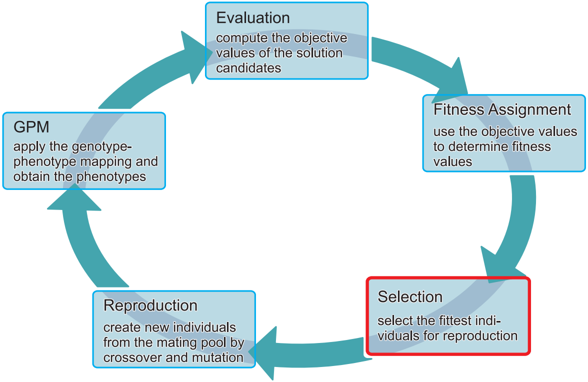
- Fitness is relative: e.g., Pareto optimal *inside population* means non-dominated by the other individuals



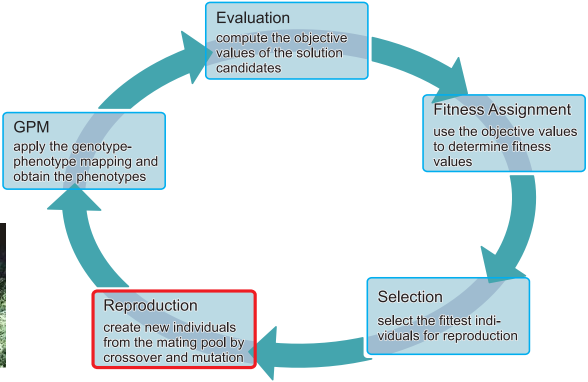


- Select the best individuals with highest probability

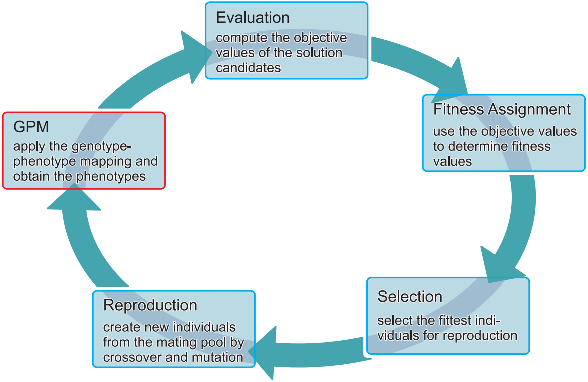




- Mutation and recombination



- Start with new population in next generation.



$\tilde{X} \longleftarrow \text{basicMOEA}(\vec{f}, ps, mps)$

begin

  $\text{pop} \longleftarrow$ create initial population

  while $\neg shouldTerminate$ **do**

    perform genotype-phenotype mapping

    compute objective values

    assign fitness

    $\text{matePool} \longleftarrow$ select parents from pop

    $\text{pop} \longleftarrow$ apply reproduction operations

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- A Multi-Objective Evolutionary Algorithm works like a normal EA
- Initialize first generation and generation counter

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- Use (e.g., traditional) selection algorithm
- Apply reproduction operators mutation and crossover

Dealing with Multiple Solutions

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- If size limit is reached: delete elements from the archive (maybe randomly, maybe based on density information).

Section Outline

1. Introduction
2. Lexicographic Optimization
3. Weighted-Sum Approach
4. Pareto-based Approach
5. MOEAs
6. Pareto Ranking
7. Problems

Pareto Ranking

- Reflect the Pareto dominance relationship of the individuals in the population in the fitness! [41, 54–56]

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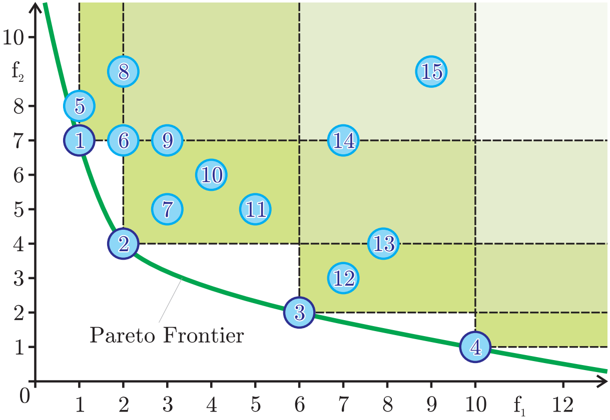
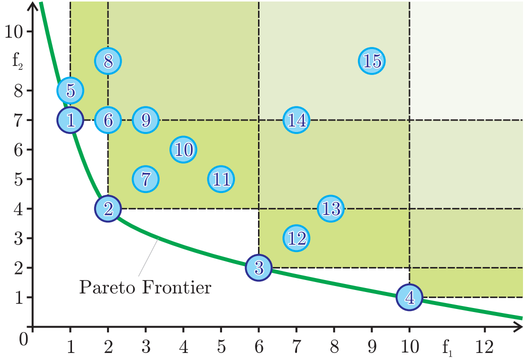


Example: two-objective problem with 15 individuals (1) to (15)

- Reflect the Pareto dominance relationship of the individuals in the population in the fitness! [41, 54–56]
- Idea: Count the number dominates(p, pop) of individuals that the individual p and set $\nu(p) = {}^{1}/_{(1+\textit{dominates}(p,\mathrm{pop}))}$



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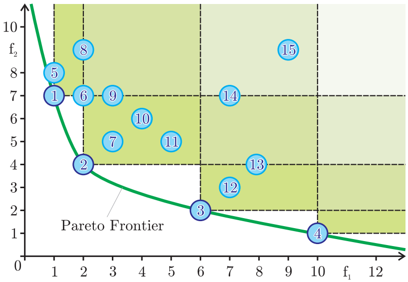

p	dominated by p	$v(p)$
1	{5, 6, 8, 9, 14, 15}	1/7
2	{6, 7, 8, 9, 10, 11, 13, 14, 15}	1/10
3	{12, 13, 14, 15}	1/5
4	∅	1
5	{8, 15}	1/3
6	{8, 9, 14, 15}	1/5
7	{9, 10, 11, 14, 15}	1/6
8	{15}	1/2
9	{14, 15}	1/3
10	{14, 15}	1/3
11	{14, 15}	1/3
12	{13, 14, 15}	1/4
13	{15}	1/2
14	{15}	1/2
15	∅	1

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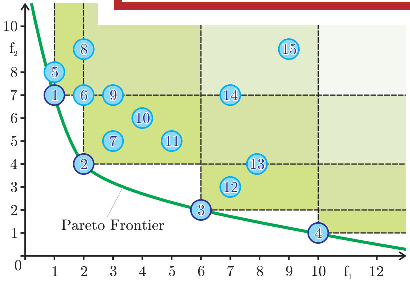

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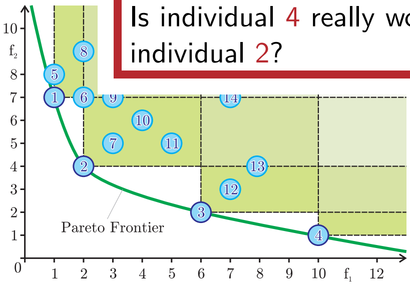
Are individual 4 and 15 really as same as interesting?
Is individual 4 really worse than individual 2?



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10	{14, 15}	1/3
11	{14, 15}	1/3
12	{13, 14, 15}	1/4
13	{15}	1/2
14	{15}	1/2
15	$\varnothing$	**1**

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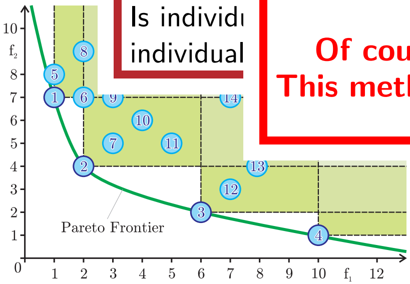
Is individu

individual

Of course not!
This method is bad.



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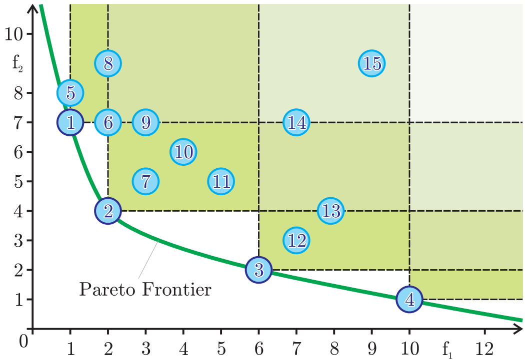
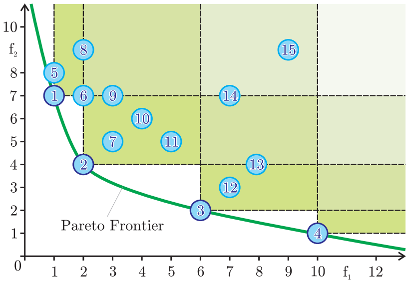
15	∅	**1**

- Reflect the Pareto dominance relationship of the individuals in the population in the fitness! [41, 54–56]
- Idea: Count the number $\#\mathrm{dom}(p, \mathrm{pop})$ of individuals in the population that dominate individual p, set $\nu(p) = \#\mathrm{dom}(p, \mathrm{pop})$



- Reflect the Pareto dominance relationship of the individuals in the population in the fitness! [41, 54–56]
- Pareto Ranking: Count the number $\#\mathrm{dom}(p, \mathrm{pop})$ of individuals in the population that dominate individual p, set $\nu(p) = \#\mathrm{dom}(p, \mathrm{pop})$



p	#dom(p, Pop)	$v(p)$
1	∅	0
2	∅	0
3	∅	0
4	∅	0
5	{1}	1
6	{1, 2}	2
7	{2}	1
8	{1, 2, 5, 6}	4
9	{1, 2, 6, 7}	4
10	{2, 7}	2
11	{2, 7}	2
12	{3}	1
13	{2, 3, 12}	3
14	{1, 2, 3, 6, 7, 9, 10, 11, 12}	9
15	{1,2,3,5,6,7,8,9,10,11,12,13,14}	13

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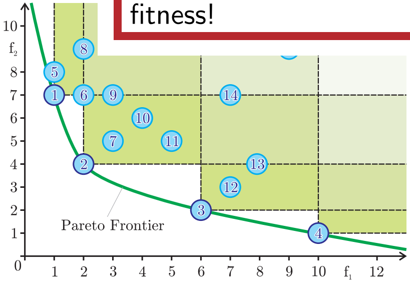
- Pareto Ranking: Count the number $\#\mathrm{dom}(p, \mathrm{pop})$ of individuals in the population that dominate individual p, se

All the non-dominated individuals have the same, best possible fitness!



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9	{1, 2, 6, 7}	4
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11	{2, 7}	2
12	{3}	1
13	{2, 3, 12}	3
14	{1, 2, 3, 6, 7, 9, 10, 11, 12}	9
15	{1,2,3,5,6,7,8,9,10,11,12,13,14}	13

Pareto Ranking

paretoRank(pop)

begin
  22**foreach** $p \in pop$ **do** $p.y \longleftarrow 0$
  for $i \longleftarrow 1$ **up to** $ps - 1$ **do**
    $p_1 \longleftarrow \mathrm{pop}[i]$
    for $j \longleftarrow 0$ **up to** $i - 1$ **do**
      $p_2 \longleftarrow \mathrm{pop}[j]$
      if $p_1 \dashv p_2$ **then** $p_2.y \longleftarrow p_2.y + 1$
      if $p_2 \dashv p_1$ **then** $p_1.y \longleftarrow p_1.y + 1$

- Pareto ranking can easily be implemented (see [56] for better method):

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```
begin
    22foreach p ∈ pop do p.y ⟵ 0

    for i ⟵ 1 up to ps − 1 do
        p1 ⟵ pop[i]
        for j ⟵ 0 up to i − 1 do
            p2 ⟵ pop[j]
            if p1 ⊣ p2 then p2.y ⟵ p2.y + 1

            if p2 ⊣ p1 then p1.y ⟵ p1.y + 1
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- Initialize fitness of all individuals to zero (Fitness is subject to minimization)

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      if $p_2 \dashv p_1$ **then** $p_1.y \longleftarrow p_1.y + 1$

- Pareto ranking can easily be implemented (see [56] for better method):
- Initialize fitness of all individuals to zero (Fitness is subject to minimization)
- Compare each individual $p_1 \ldots$

paretoRank(pop)

```
begin
    22foreach p ∈ pop do p.y ⟵ 0

    for i ⟵ 1 up to ps − 1 do
        p₁ ⟵ pop[i]
        for j ⟵ 0 up to i − 1 do
            p₂ ⟵ pop[j]
            if p₁ ⊣ p₂ then p₂.y ⟵ p₂.y + 1

            if p₂ ⊣ p₁ then p₁.y ⟵ p₁.y + 1
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- Pareto ranking can easily be implemented (see [56] for better method):
- Compare each individual p_1...
- ...with each other individual p_2

paretoRank(pop)

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    22foreach p ∈ pop do p.y ⟵ 0

    for i ⟵ 1 up to ps − 1 do
        p1 ⟵ pop[i]
        for j ⟵ 0 up to i − 1 do
            p2 ⟵ pop[j]
            if p1 ⊣ p2 then p2.y ⟵ p2.y + 1

            if p2 ⊣ p1 then p1.y ⟵ p1.y + 1
```

- Pareto ranking can easily be implemented (see [56] for better method):
- . . . with each other individual p_2
- If p_1 wins, count it as loss for p_2

paretoRank(pop)

begin
  22**foreach** $p \in pop$ **do** $p.y \longleftarrow 0$

  for $i \longleftarrow 1$ **up to** $ps - 1$ **do**
    $p_1 \longleftarrow \mathrm{pop}[i]$
    for $j \longleftarrow 0$ **up to** $i - 1$ **do**
      $p_2 \longleftarrow \mathrm{pop}[j]$
      if $p_1 \dashv p_2$ **then** $p_2.y \longleftarrow p_2.y + 1$

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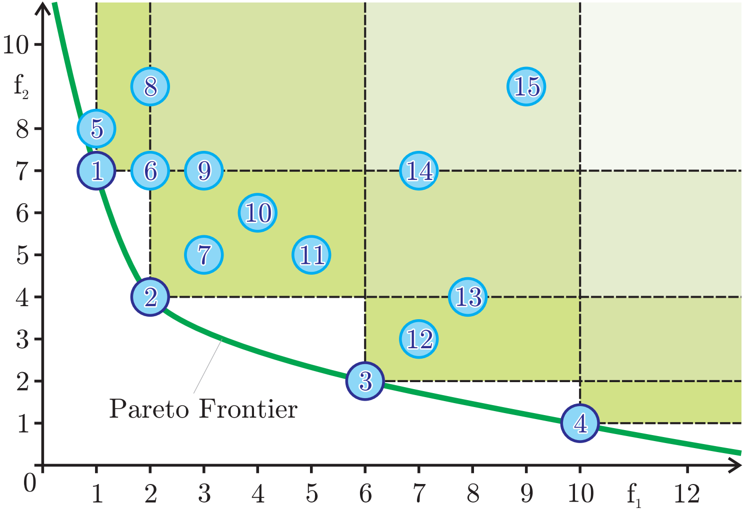
Section Outline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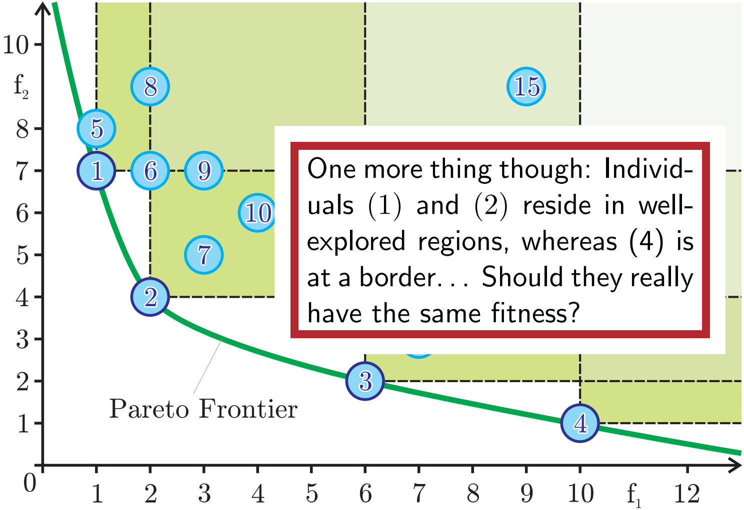
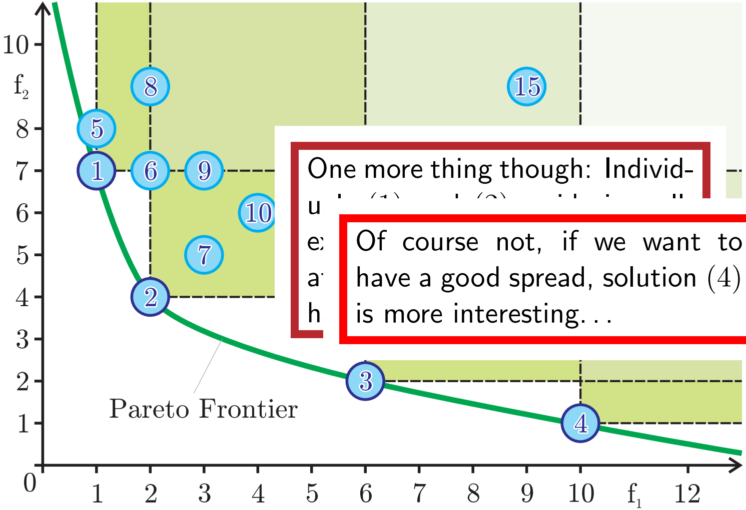
Pareto Frontier



One more thing though: Individuals (1) and (2) reside in well-explored regions, whereas (4) is at a border... Should they really have the same fitness?



One more thing though: Individ-
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Of course not, if we want to have a good spread, solution (4) is more interesting...

Practical MOEAs

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  3. PESA by Corne et al. [59] (Pareto domination and number of other individuals in the same hyper-box in a grid defined over the search space)

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  5. If population not empty, go to step 1.

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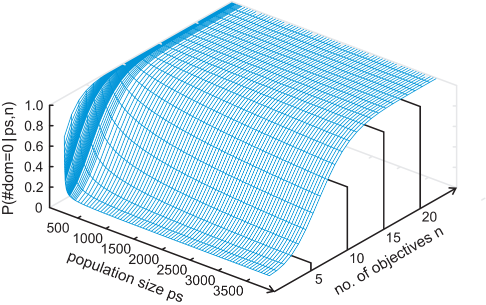

Fraction of non-dominated elements in ps uniform random samples of dimension n

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  3. The number of possible Pareto-optimal solutions may increase exponentially.
- Therefore: Do NOT use too many objective functions!

Summary

- Many problems are multi-objective by nature, for example Genetic Programming: minimize Symbolic Regression expression size and error

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- If objectives conflict, a special treatment is necessary
- Pareto ranking is a good idea: incorporated into MOEAs
- Many treatments (e.g., Pareto) can be reduced to binary comparisons

谢谢

Thank you

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Caspar David Friedrich, “Der Wanderer über dem Nebelmeer”, 1818
http://en.wikipedia.org/wiki/Wanderer_above_the_Sea_of_Fog

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