合肥学院
HEFEI UNIVERSITY

Metaheuristic Optimization

8. Tabu Search

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Outline

1. Introduction
2. Tabu Search
3. Example 1: MAX-SAT
4. Example 2: Traveling Salesman Problem
5. Iterated Local Search
6. Summary

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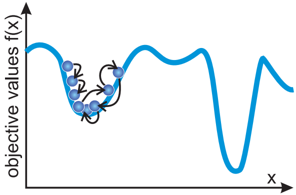

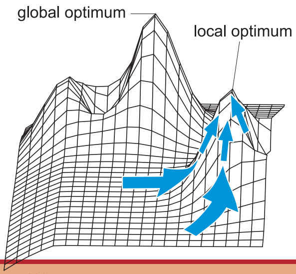

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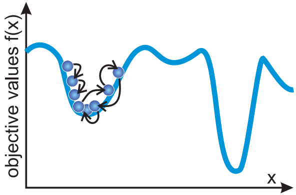

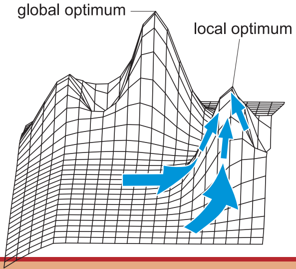

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- A “neighboring solution” is a solution which can be reached with a single application of a unary search operation.
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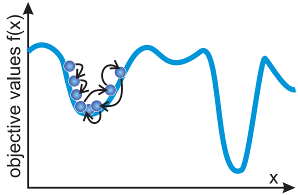

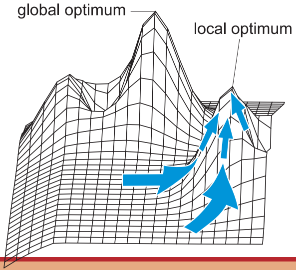

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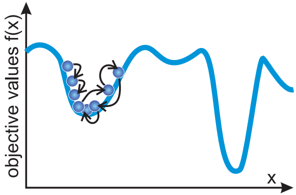

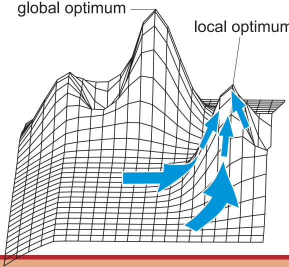

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- Hill climbers will get stuck at *any* optimum, because they will only move from one solution to a better solution.
- They are likely to converge to local optimum, i.e., may not give us the globally optimal solution.
- Simulated Annealing can avoid this, because it sometimes (proabilistically) also accepts worse candidate solutions.
- Tabu Search, introduced by Glover, Glover [1, 2], is another local search which introduces another, similar approach.

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- It will pick this solution even if it is worse than the current solution.
- Problem: This can easily lead to cycles (if the current solution is a local optimum, the search will go to a worse solution and then immediately back to the previous one, the local optimum).
- Solution: Introduce a *tabu criterion* which forbids certain solutions to be visited, to avoid re-visiting already seen solutions.

Tabu List

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- Store features of tt most recently visited solutions tt is called *tabu tenure* or *tabu list length*).
- Solutions with features from the tabu list are forbidden.
- Choice of tt has big influence on performance.

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- Tabu criterion may also prevent previously unseen solutions from being explored.
- Some of these might be better than the best solution we have found so far, i.e., very interesting regardless whether they are tabu or not…
- *Aspiration criteria*: criteria that override the tabu criterion and allow the search to move to a solution even if it is tabu.

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

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begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
              ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
            then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
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            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
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- We assume a simple Tabu Search where

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begin
- $p_{best}.x \longleftarrow$ create initial solution
- $p_{best}.y \longleftarrow f(p_{best}.x)$
- $p_{cur}.y \longleftarrow p_{best}$
- $tabu \longleftarrow$ empty list
- **while** $\neg(shouldTerminate \vee (p_{cur} \neq \emptyset))$ **do**
  - $p_{new} \longleftarrow \emptyset$
  - **foreach** $p_{test} \in$ neighborhood of p_{cur} **do**
    - $p_{test}.y \longleftarrow f(p_{test}.x)$
    - **if** $((move \notin tabu) \wedge ((p_{new} = \emptyset) \vee (p_{test}.y < p_{new}.y))) \vee (p_{test}.y \leq p_{best}.y)$ **then**
      - $p_{new} \longleftarrow p_{test}$
      - $move_b \longleftarrow move$
  - $p_{cur} \longleftarrow p_{new}$
  - **if** $(p_{cur} \neq \emptyset)$ **then**
    - **if** $p_{cur}.y \leq p_{best}.y$ **then** $p_{best} \longleftarrow p_{cur}$
    - append $\overline{move_b}$ to $tabu$
    - **if** length of $tabu \geq tt$ **then** remove oldest element from $tabu$
- **return** p_{best}

- We assume a simple Tabu Search where
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  - where the tabu criterion is the applied search move

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- We assume a simple Tabu Search where
  - search- and solution space are the same ($\mathbb{G} = \mathbb{X}$) and
  - where the tabu criterion is the applied search move and
  - where the aspiration criterion is that any solution better than best currently known solution p_{best} will always be accepted

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- We start by creating the starting point of our search (here directly in form of candidate solution $p_{cur}.x$).

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             then
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        if (p_cur ≠ ∅) then
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            append overline(move_b) to tabu
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- We start by creating the starting point of our search (here directly in form of candidate solution $p_{cur}.x$).
- This could happen randomly or via a simple logic (constructive heuristic)

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- We compute the objective value $f(p_{cur}.x)$ of the initial solution and remember it in variable $p_{cur}.y$.

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- The initial solution p_{cur} is also the best solution p_{best} we know so far.

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- Initially, the tabu list $tabu$ is empty, everything is allowed.

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Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
              ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
            then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- In every iteration, we first check the termination criterion whether we should quit.

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
  p_best.x ⟵ create initial solution
  p_best.y ⟵ f(p_best.x)
  p_cur.y ⟵ p_best
  tabu ⟵ empty list
  while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
    p_new ⟵ ∅
    foreach p_test ∈ neighborhood of p_cur do
      p_test.y ⟵ f(p_test.x)
      if
        ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
        (p_test.y ≤ p_best.y)
      then
        p_new ⟵ p_test
        move_b ⟵ move
    p_cur ⟵ p_new
    if (p_cur ≠ ∅) then
      if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

      append ‾move_b‾ to tabu
      if length of tabu ≥ tt then remove oldest element from tabu
  return p_best
```

- We should also stop if all solutions surrounding our current solution are tabu and the aspiration criterion does not hold for any, i.e., if there is no next solution to move to.

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

begin
- $p_{best}.x \longleftarrow$ create initial solution
- $p_{best}.y \longleftarrow f(p_{best}.x)$
- $p_{cur}.y \longleftarrow p_{best}$
- $tabu \longleftarrow$ empty list
- **while** $\neg\,(shouldTerminate \lor (p_{cur} \neq \emptyset))$ **do**
  - $p_{new} \longleftarrow \emptyset$
  - **foreach** $p_{test} \in$ neighborhood of p_{cur} **do**
    - $p_{test}.y \longleftarrow f(p_{test}.x)$
    - **if** $((move \notin tabu) \land ((p_{new} = \emptyset) \lor (p_{test}.y < p_{new}.y)))\lor$ $(p_{test}.y \leq p_{best}.y)$ **then**
      - $p_{new} \longleftarrow p_{test}$
      - $move_b \longleftarrow move$
  - $p_{cur} \longleftarrow p_{new}$
  - **if** $(p_{cur} \neq \emptyset)$ **then**
    - **if** $p_{cur}.y \leq p_{best}.y$ **then** $p_{best} \longleftarrow p_{cur}$
    - append $\overline{move_b}$ to $tabu$
    - **if** length of $tabu \geq tt$ **then** remove oldest element from $tabu$
- **return** p_{best}

- In each step, we first assume that there is no solution p_{new} we can move to from p_{cur}.

Putting it Together

$p_{best} \longleftarrow \mathrm{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
              ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
             then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- We then scan the complete neighborhood of p_{cur}.

Putting it Together

$p_{best} \longleftarrow \mathrm{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] $\mathrm{shouldTerminate}$: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
  p_best.x ⟵ create initial solution
  p_best.y ⟵ f(p_best.x)
  p_cur.y ⟵ p_best
  tabu ⟵ empty list
  while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
    p_new ⟵ ∅
    foreach p_test ∈ neighborhood of p_cur do
      p_test.y ⟵ f(p_test.x)
      if
        ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
        (p_test.y ≤ p_best.y)
      then
        p_new ⟵ p_test
        move_b ⟵ move
    p_cur ⟵ p_new
    if (p_cur ≠ ∅) then
      if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

      append ‾move_b‾ to tabu
      if length of tabu ≥ tt then remove oldest element from tabu
  return p_best
```

- We then scan the complete neighborhood of p_{cur}.
- This neighborhood is defined by possible search moves $move$ that can be applied to the current candidate solution $p_{cur}.x$ (again, here we assume that $\mathbb{G} = \mathbb{X}$).

Putting it Together

$p_{best} \longleftarrow \mathrm{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] $\mathrm{shouldTerminate}$: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
              ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
            then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- We then scan the complete neighborhood of p_{cur}.
- This neighborhood is defined by possible search moves $move$ that can be applied to the current candidate solution $p_{cur}.x$ (again, here we assume that $\mathbb{G} = \mathbb{X}$).
- For example, if our candidate solutions are strings of n bits, a neighborhood could be any string that can be reached by flipping a single bit in $p_{cur}.x$ (and this neighborhood would contain n other solutions $p_{test}.x$).

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
              ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
            then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- We compute the objective value $f(p_{test}.x)$ of the initial solution and remember it in variable $p_{test}.y$.

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

begin
- $p_{best}.x \longleftarrow$ create initial solution
- $p_{best}.y \longleftarrow f(p_{best}.x)$
- $p_{cur}.y \longleftarrow p_{best}$
- $tabu \longleftarrow$ empty list
- **while** $\neg(shouldTerminate \vee (p_{cur} \neq \emptyset))$ **do**
  - $p_{new} \longleftarrow \emptyset$
  - **foreach** $p_{test} \in$ neighborhood of p_{cur} **do**
    - $p_{test}.y \longleftarrow f(p_{test}.x)$
    - **if** $((move \notin tabu) \wedge ((p_{new} = \emptyset) \vee (p_{test}.y < p_{new}.y))) \vee$ $(p_{test}.y \leq p_{best}.y)$ **then**
      - $p_{new} \longleftarrow p_{test}$
      - $move_b \longleftarrow move$
  - $p_{cur} \longleftarrow p_{new}$
  - **if** $(p_{cur} \neq \emptyset)$ **then**
    - **if** $p_{cur}.y \leq p_{best}.y$ **then** $p_{best} \longleftarrow p_{cur}$
    - append $\overline{move_b}$ to $tabu$
    - **if** length of $tabu \geq tt$ **then** remove oldest element from $tabu$
- **return** p_{best}

- p_{test} would be a candidate for the next step of our search

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
              ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
             then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- p_{test} would be a candidate for the next step of our search if and only if
    1. the move $move$ leading to it from p_{cur} is not tabu

Putting it Together

$p_{best} \longleftarrow \mathrm{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] $\mathrm{shouldTerminate}$: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
             ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
             (p_test.y ≤ p_best.y)
            then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- p_{test} would be a candidate for the next step of our search if and only if
  1. the move $move$ leading to it from p_{cur} is not tabu and
     1. it is better than the currently best acceptable neighbor p_{new}

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
  p_best.x ⟵ create initial solution
  p_best.y ⟵ f(p_best.x)
  p_cur.y ⟵ p_best
  tabu ⟵ empty list
  while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
    p_new ⟵ ∅
    foreach p_test ∈ neighborhood of p_cur do
      p_test.y ⟵ f(p_test.x)
      if
       ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
       (p_test.y ≤ p_best.y)
      then
        p_new ⟵ p_test
        move_b ⟵ move
    p_cur ⟵ p_new
    if (p_cur ≠ ∅) then
      if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

      append ‾move_b‾ to tabu
      if length of tabu ≥ tt then remove oldest element from tabu
  return p_best
```

- p_{test} would be a candidate for the next step of our search if and only if
  1. the move $move$ leading to it from p_{cur} is not tabu and
     1. it is better than the currently best acceptable neighbor p_{new} or
     2. it is the first acceptable neighbor.

Putting it Together

$p_{best} \longleftarrow \mathrm{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] $\mathrm{shouldTerminate}$: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
              ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
              then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- p_{test} would be a candidate for the next step of our search if and only if
  1. the move $move$ leading to it from p_{cur} is not tabu and
     1. it is better than the currently best acceptable neighbor p_{new} or
     2. it is the first acceptable neighbor.
  2. or the aspiration criterion kicks in, which here means that it is better than the best solution p_{best} we have ever seen.

Putting it Together

$p_{best} \longleftarrow \mathrm{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] $\mathrm{shouldTerminate}$: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
              ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
             then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- In this case, we
  - remember it in variable p_{new}

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

begin
- $p_{best}.x \longleftarrow$ create initial solution
- $p_{best}.y \longleftarrow f(p_{best}.x)$
- $p_{cur}.y \longleftarrow p_{best}$
- $tabu \longleftarrow$ empty list
- **while** $\neg\,(shouldTerminate \vee (p_{cur} \neq \emptyset))$ **do**
  - $p_{new} \longleftarrow \emptyset$
  - **foreach** $p_{test} \in$ neighborhood of p_{cur} **do**
    - $p_{test}.y \longleftarrow f(p_{test}.x)$
    - **if** $((move \notin tabu) \wedge ((p_{new} = \emptyset) \vee (p_{test}.y < p_{new}.y))) \vee (p_{test}.y \leq p_{best}.y)$ **then**
      - $p_{new} \longleftarrow p_{test}$
      - $move_b \longleftarrow move$
  - $p_{cur} \longleftarrow p_{new}$
  - **if** $(p_{cur} \neq \emptyset)$ **then**
    - **if** $p_{cur}.y \leq p_{best}.y$ **then** $p_{best} \longleftarrow p_{cur}$
    - append $\overline{move_b}$ to $tabu$
    - **if** length of $tabu \geq tt$ **then** remove oldest element from $tabu$
- **return** p_{best}

- In this case, we
  - remember it in variable p_{new} and
  - store the $move$ leading to it (coming from p_{cur}) in variable $move_b$.

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
              ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
             then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- After we have scanned the whole neighborhood of p_{cur}, we store the best discovered acceptable solution p_{new} in p_{cur}. (This could also be nothing $\emptyset$...)

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
              ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
             then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- If we actually found new acceptable point p_{cur}

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
              ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
            then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append ¯move_b to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- If we actually found new acceptable point p_{cur}
  - We check if it is better than the best solution p_{best} we have ever found and, if so, store it in p_{best}.

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
Data: p_{cur}: the current solution
Data: $move$: the $move$ reaching p_{test}
Data: $move_b$: the $move$ reaching p_{new}
Output: p_{best}: the best individual ever discovered

```
begin
    p_best.x ⟵ create initial solution
    p_best.y ⟵ f(p_best.x)
    p_cur.y ⟵ p_best
    tabu ⟵ empty list
    while ¬(shouldTerminate ∨ (p_cur ≠ ∅)) do
        p_new ⟵ ∅
        foreach p_test ∈ neighborhood of p_cur do
            p_test.y ⟵ f(p_test.x)
            if
             ((move ∉ tabu) ∧ ((p_new = ∅) ∨ (p_test.y < p_new.y)))∨
              (p_test.y ≤ p_best.y)
             then
                p_new ⟵ p_test
                move_b ⟵ move
        p_cur ⟵ p_new
        if (p_cur ≠ ∅) then
            if p_cur.y ≤ p_best.y then p_best ⟵ p_cur

            append overline(move_b) to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- If we actually found new acceptable point p_{cur}
  - We store the inverse $\overline{move_b}$ of the move $move_b$ leading from the “old” p_{cur} to the “new” p_{cur} in the tabu list $tabu$ to prevent us from going back to the “old” p_{cur} in the next tt iterations.

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
Data: p_{new}: the new solution to be tested
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  - We store the inverse $\overline{move}_b$ of the move $move_b$ leading from the "old" p_{cur} to the "new" p_{cur} in the tabu list $tabu$ to prevent us from going back to the "old" p_{cur} in the next tt iterations.
  - If the tabu list $tabu$ is now longer than the tabu tenure tt, we delete the oldest element from it.

Putting it Together

$p_{best} \longleftarrow \text{tabuSearch}(f, tt)$

Input: f: the objective function subject to minization
Input: [implicit] shouldTerminate: the termination criterion
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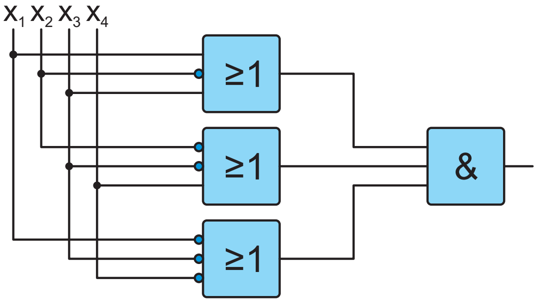
            append ‾move_b‾ to tabu
            if length of tabu ≥ tt then remove oldest element from tabu
    return p_best
```

- Finally, if we have met the termination criterion shouldTerminate or there simply is no acceptable solution to go to anymore, we return the best solution p_{best} we found so far.

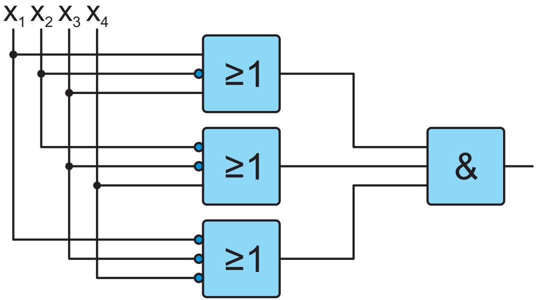
Section Outline

1. Introduction
2. Tabu Search
3. Example 1: MAX-SAT
4. Example 2: Traveling Salesman Problem
5. Iterated Local Search
6. Summary

- Satisfiability Problems (SAT) [3]



- Satisfiability Problems (SAT) [3]:
  - Given: Formula B in Boolean logic



- Satisfiability Problems (SAT) [3]:
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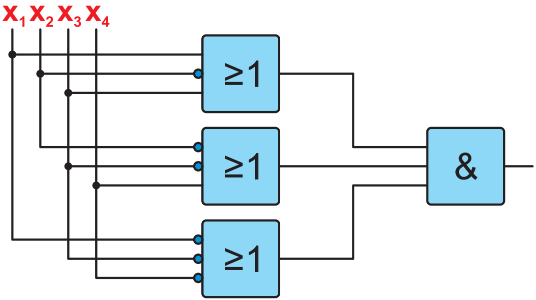

- Satisfiability Problems (SAT) [3]:
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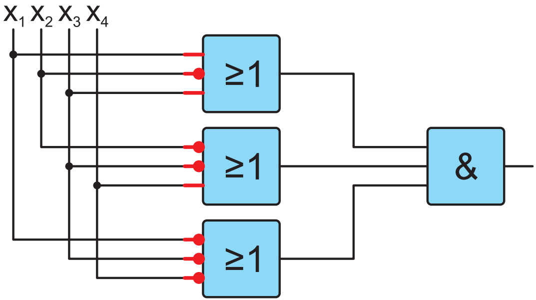

- Satisfiability Problems (SAT) [3]:
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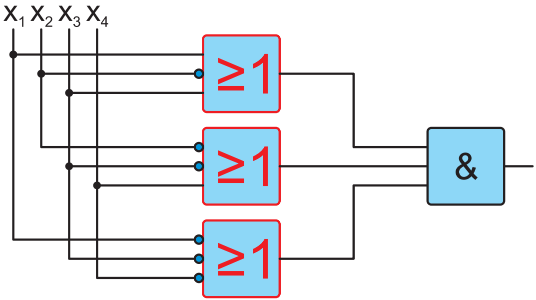

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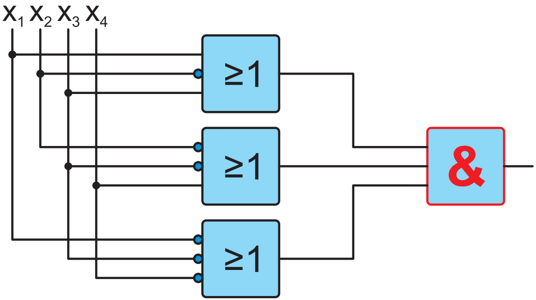

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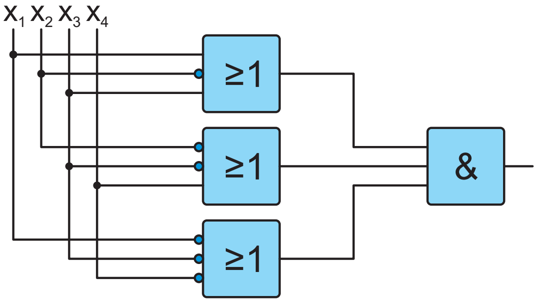
  - Given: Formula B in Boolean logic with of n Boolean variables $\vec{x} = (x_1, x_2, \dots, x_n)$, which appear either directly or negated in k "or" clauses, which are all combined with into one "and"
  - SAT Goal: find a setting for these variables so that B becomes true



- Maximum Satisfiability Problems (SAT) [4]:
  - Given: Formula B in Boolean logic with of n Boolean variables $\vec{x} = (x_1, x_2, \dots, x_n)$, which appear either directly or negated in k "or" clauses, which are all combined with into one "and"
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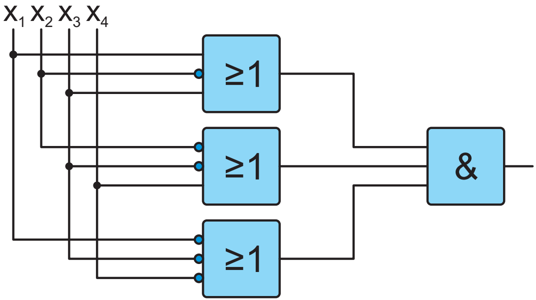

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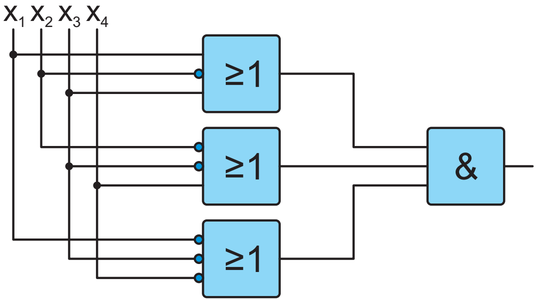

Example: MAX-SAT

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- Candidate solution: string of n bits.

Tabu Search for MAX-SAT

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Tabu Search for MAX-SAT

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- Neighborhood of candidate solution x: other bit strings assignments which differ in exactly one bit
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- Aspiration criterion: if flipping the variable would lead to a new best-so-far solution, we will accept it even if it is tabu

Section Outline

1 Introduction

2 Tabu Search

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6 Summary

- Example: Traveling Salesman Problem (TSP)

- Example: Traveling Salesman Problem (TSP): Find a cyclic path of minimal costs that visits a set of cities V [5–8]

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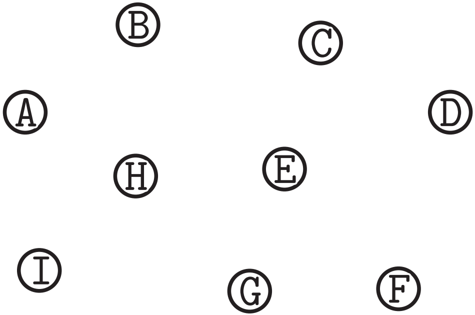

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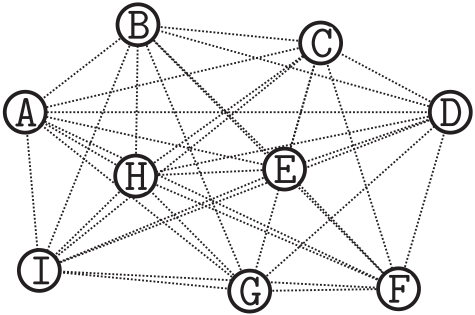

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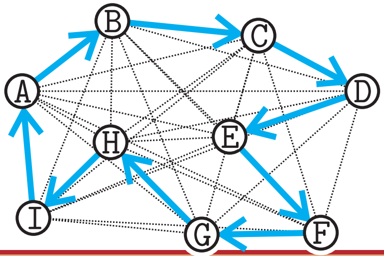

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Neighborhood 1: Swap Operator

- $\texttt{swap}(\mathbf{x}, i, j)$: swap the element at index i in permutation $\mathbf{x}$ with element at index j [9–14]

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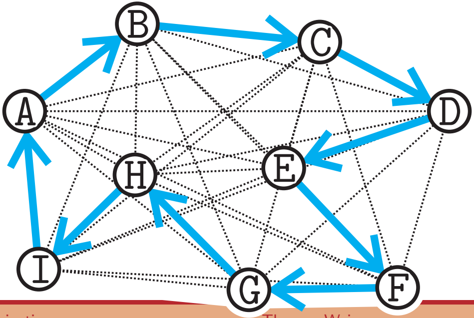

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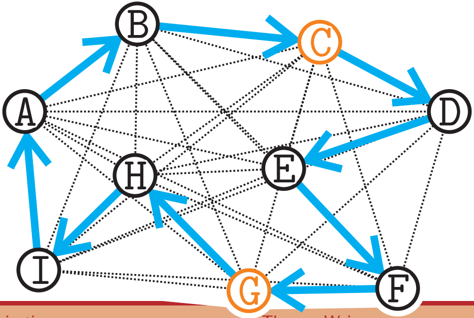
- $\mathbf{x} = ($A,B,C,D,E,F,G,H,I$)$
- swap($\mathbf{x}, 3, 7$)



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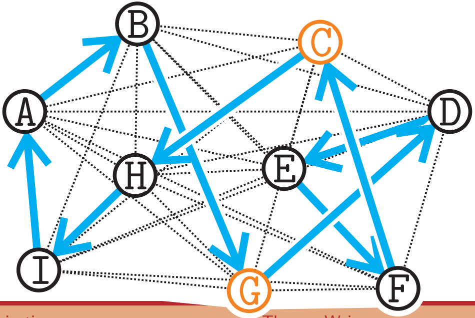

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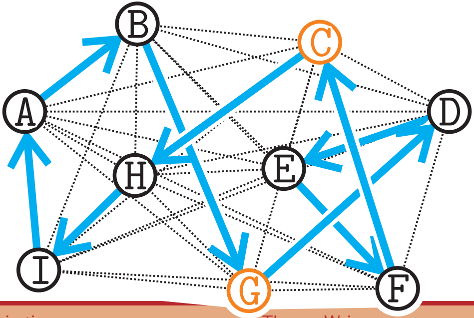
- Possible 4-opt move



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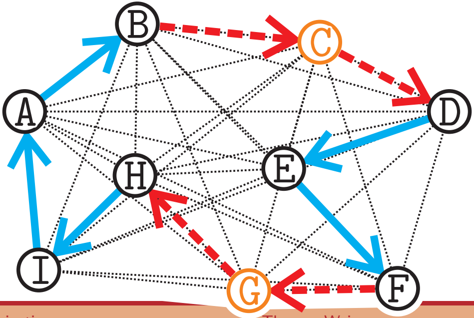
- Possible 4-opt move: delete four edges



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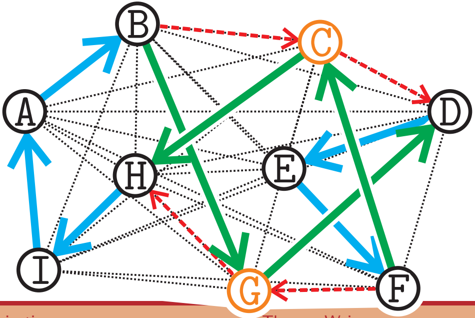
- Possible 4-opt move: delete four edges and add four edges



- `reverse`$(\mathbf{x}, i, j)$: reverse the subsequence between indexes i and j in permutation $\mathbf{x}$ [9, 15–19]

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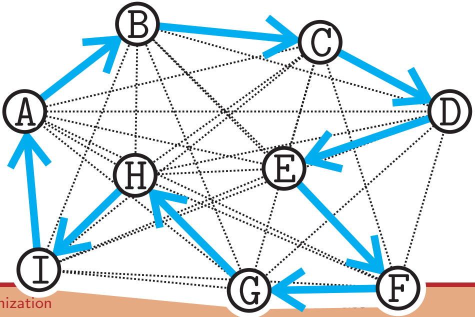

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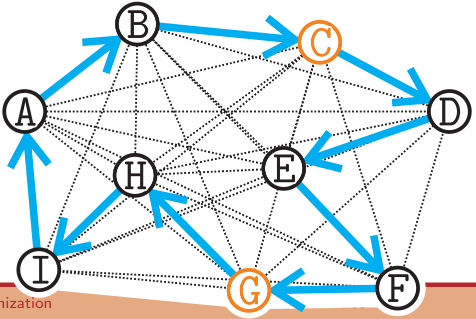
- $\mathbf{x} = (\texttt{A},\texttt{B},\texttt{C},\texttt{D},\texttt{E},\texttt{F},\texttt{G},\texttt{H},\texttt{I})$
- $\texttt{reverse}_1(\mathbf{x}, 3, 7)$



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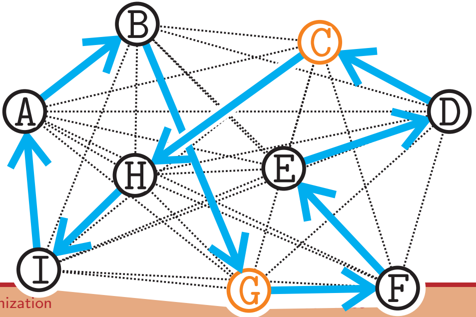

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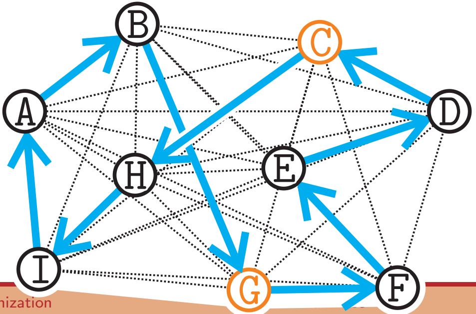
- Possible 2-opt move [19–21]



- $\texttt{reverse}(\mathbf{x}, i, j)$: reverse the subsequence between indexes i and j in permutation $\mathbf{x}$ [9, 15–19]
- $\mathbf{x} = (\texttt{A,B,C,D,E,F,G,H,I})$
- $\texttt{reverse}_1(\mathbf{x}, 3, 7) = (\texttt{A,B,}\underline{\texttt{G,F,E,D,C}}\texttt{,H,I})$
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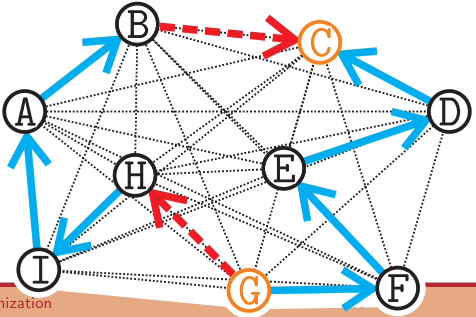

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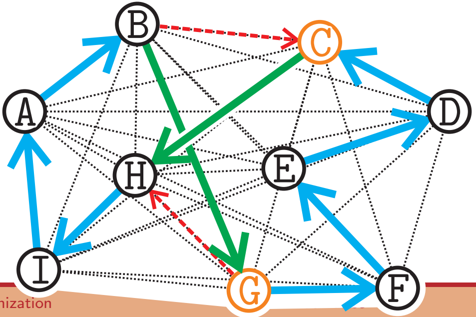
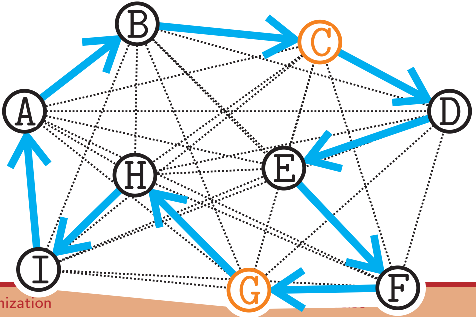


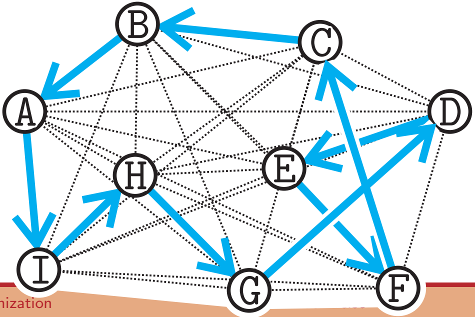
Neighborhood 2: Reverse Operator

- $\texttt{reverse}(\mathbf{x}, i, j)$: Two ways to reverse the subsequence between indexes i and j in permutation $\mathbf{x}$ [9, 15–19]
-

- $\texttt{reverse}(\mathbf{x}, i, j)$: Two ways to reverse the subsequence between indexes i and j in permutation $\mathbf{x}$ [9, 15–19]
- $\mathbf{x} = (\texttt{A,B,C,D,E,F,G,H,I})$
- $\texttt{reverse}_2(\mathbf{x}, 3, 7)$



- reverse($\mathbf{x}, i, j$): Two ways to reverse the subsequence between indexes i and j in permutation $\mathbf{x}$ [9, 15–19]
- $\mathbf{x} = (\texttt{A},\texttt{B},\texttt{C},\texttt{D},\texttt{E},\texttt{F},\texttt{G},\texttt{H},\texttt{I})$
- $\texttt{reverse}_2(\mathbf{x}, 3, 7) = (\underline{\texttt{I},\texttt{H},\texttt{G}},\texttt{D},\texttt{E},\texttt{F},\underline{\texttt{C},\texttt{B},\texttt{A}})$



- $\texttt{reverse}(\mathbf{x}, i, j)$: Two ways to reverse the subsequence between indexes i and j in permutation $\mathbf{x}$ [9, 15–19]
- $\mathbf{x} = (\texttt{A,B,C,D,E,F,G,H,I})$
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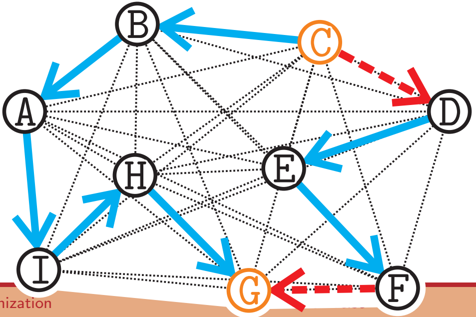

- `reverse`$(\mathbf{x}, i, j)$: Two ways to reverse the subsequence between indexes i and j in permutation $\mathbf{x}$ [9, 15–19]
- $\mathbf{x} = ($A,B,C,D,E,F,G,H,I$)$
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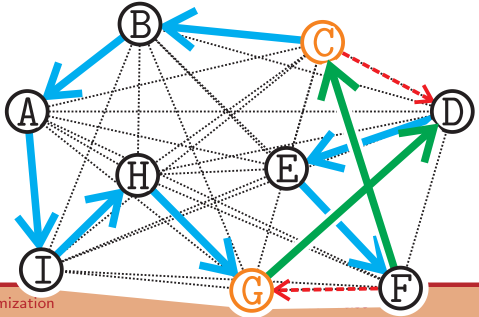

- `rotateLeft`$(\mathbf{x}, i, j)$: rotate the subsequence between indexes i and j in permutation $\mathbf{x}$ one step to the *left* [9, 11, 14, 22]

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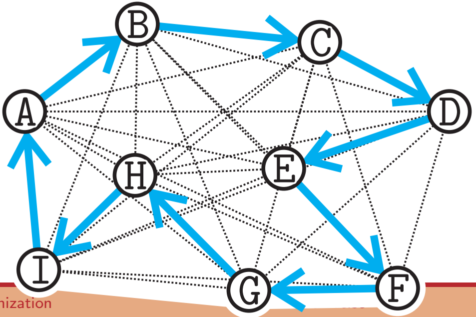

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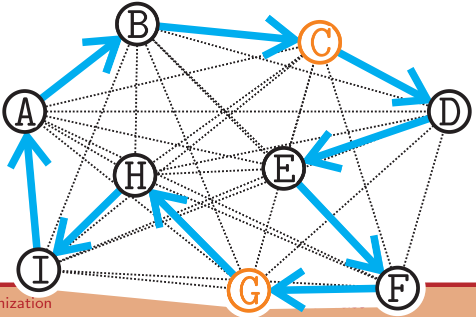
- `rotateLeft`$_1(\mathbf{x}, 3, 7)$



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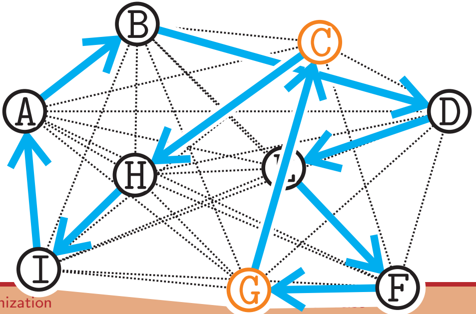
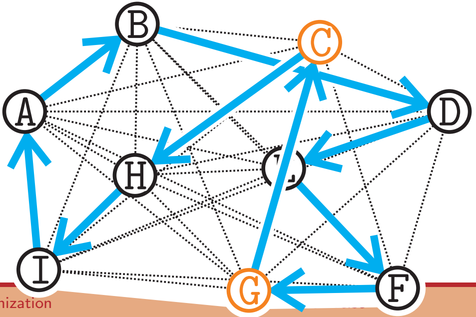


Neighborhood 3: Rotate-Left Operator

- $\texttt{rotateLeft}(\mathbf{x}, i, j)$: rotate the subsequence between indexes i and j in permutation $\mathbf{x}$ one step to the *left* [9, 11, 14, 22]
- $(\texttt{A,B,C,D,E,F,G,H,I})$
- $\texttt{rotateLeft}_1(\mathbf{x}, 3, 7) = (\texttt{A,B,}\underline{\texttt{D,E,F,G,C}}\texttt{,H,I})$
- Possible 3-opt move



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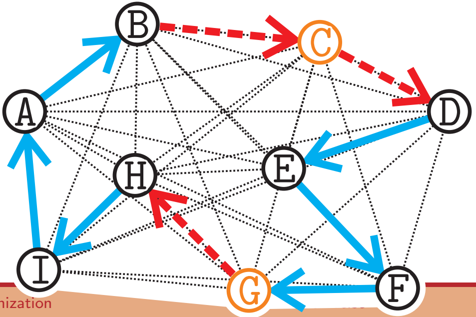

- $\texttt{rotateLeft}(\mathbf{x}, i, j)$: rotate the subsequence between indexes i and j in permutation $\mathbf{x}$ one step to the *left* [9, 11, 14, 22]
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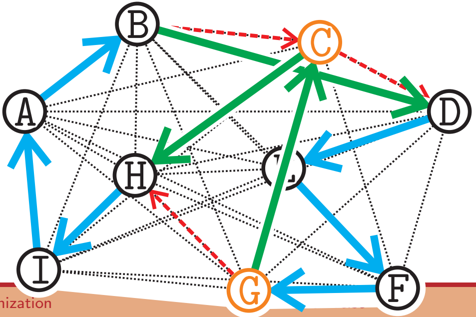

- `rotateLeft`$(\mathbf{x}, i, j)$: Two ways to rotate the subsequence between indexes i and j in permutation $\mathbf{x}$ one step to the *left* [9, 11, 14, 22]
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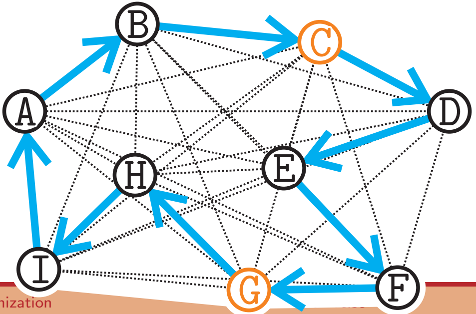
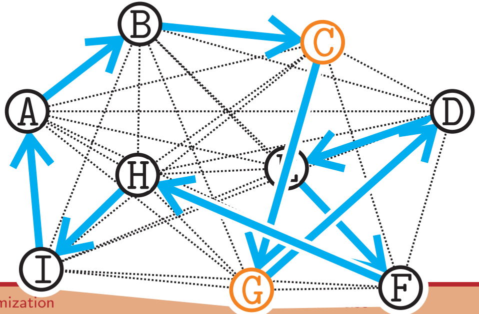


- `rotateLeft`($\mathbf{x}, i, j$): Two ways to rotate the subsequence between indexes i and j in permutation $\mathbf{x}$ one step to the *left* [9, 11, 14, 22]
- $\mathbf{x} = (\mathtt{A,B,C,D,E,F,G,H,I})$
- $\mathtt{rotateLeft}_2(\mathbf{x}, 3, 7) = (\underline{\mathtt{B,C,G}},\mathtt{D,E,F},\underline{\mathtt{H,I,A}})$



Neighborhood 4: Rotate-Right Operator

- $\texttt{rotateRight}(\mathbf{x}, i, j)$: rotate the subsequence between indexes i and j in permutation $\mathbf{x}$ one step to the *right* [9, 11, 14, 22]

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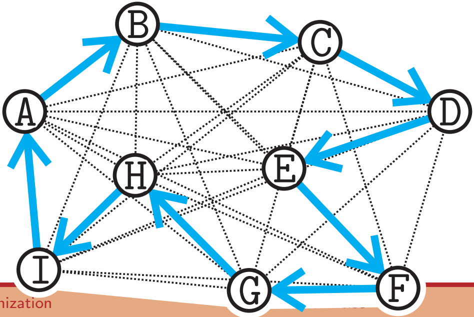

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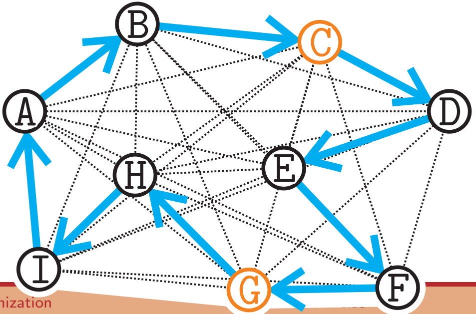
- $\texttt{rotateRight}_1(\mathbf{x}, 3, 7)$



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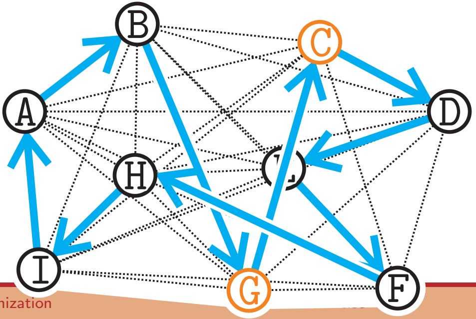

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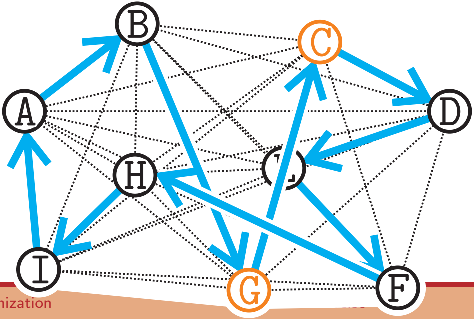

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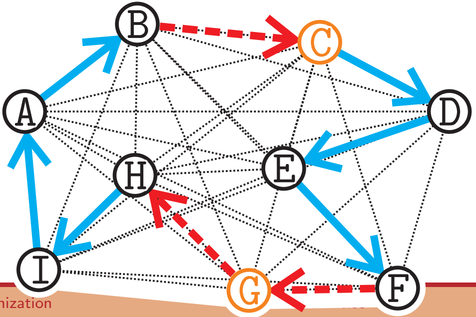

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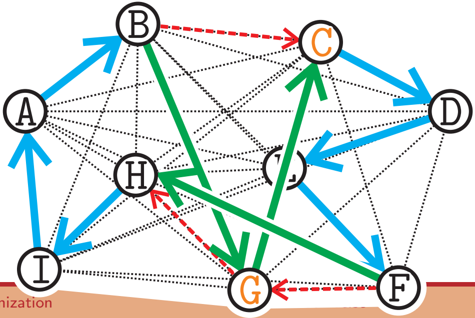

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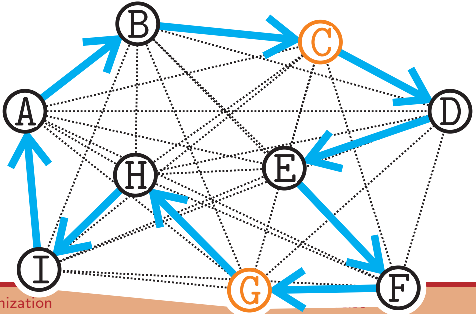
-



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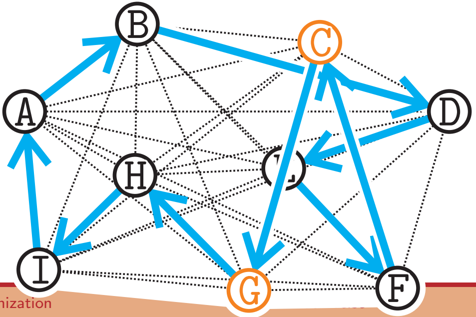
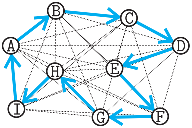
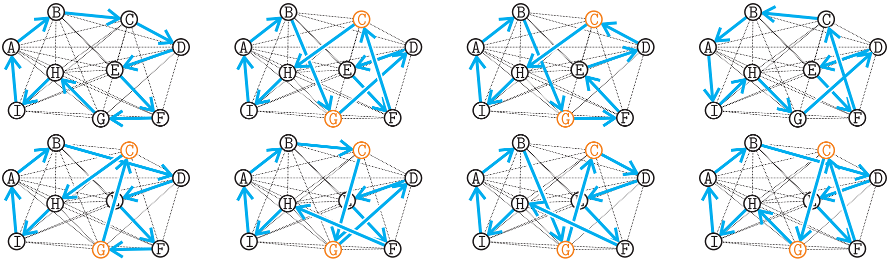


- For a candidate solution $\mathbf{x}$



Neighborhoods Summary

- For a candidate solution $\mathbf{x}$ and an index tuple (i, j), we have learned that there are seven modification operations



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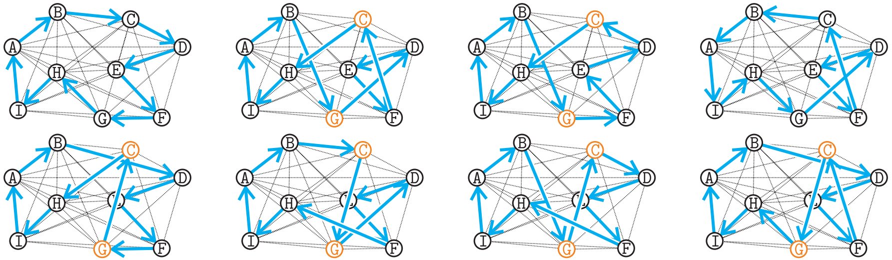

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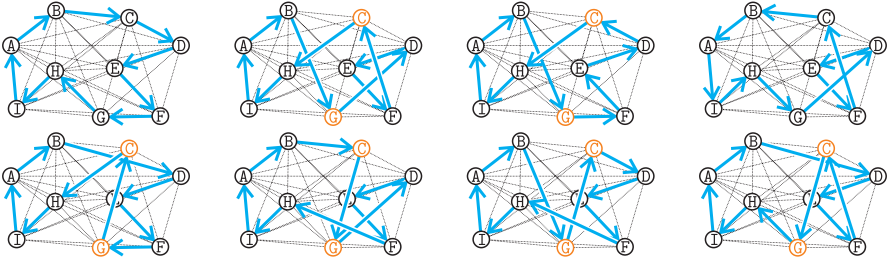
$$\Delta f = f(\mathbf{x}') - f(\mathbf{x})$$



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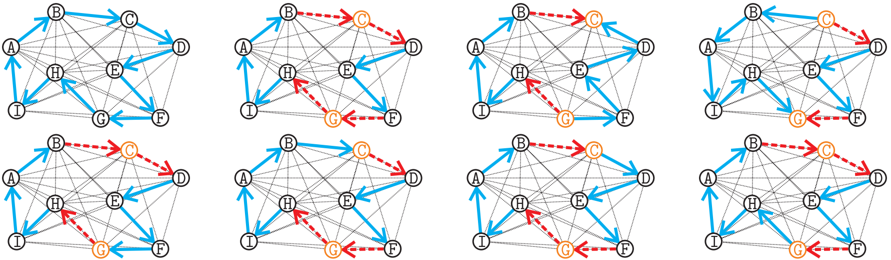

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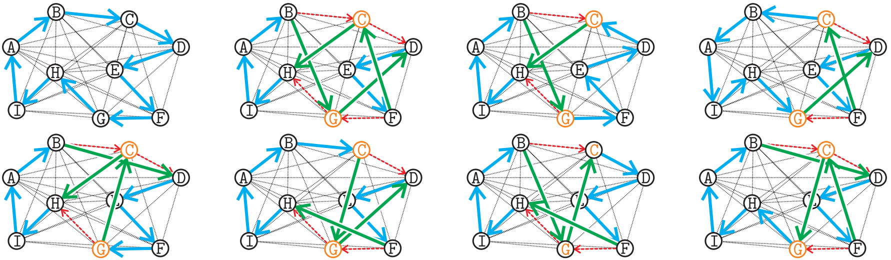
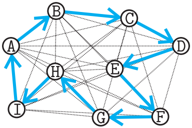
$$\Delta f = f(\mathbf{x}') - f(\mathbf{x}) = -\texttt{cost}(\text{deleted edges}) + \texttt{cost}(\text{added edges})$$



Neighborhoods Summary

- For a candidate solution $\mathbf{x}$ and an index tuple (i, j), we have learned that there are seven modification operations
- We always can compute $f(\mathbf{x}')$ in $\mathcal{O}(1)$:
- So if we choose one of these neighborhoods for our Tabu Search, we can scan the neighborhood of a solution by testing all indices i, j and for each neighbor (which is in $\mathcal{O}\left(n_v^2\right)$), we get the corresponding tour length/objective value basically for free...



- Let us consider a Tabu Search method for the Traveling Salesman Problem.

Tabu Search for the TSP

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- We can choose one of the four discussed neighborhoods, or even multiple neighborhoods.

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- Many ideas are possible. . .
- Aspiration criterion: if the new tour would be a new best-so-far solution, we will accept it even if it is tabu

1 Introduction

2 Tabu Search

3 Example 1: MAX-SAT

4 Example 2: Traveling Salesman Problem

5 Iterated Local Search

6 Summary

Iterated Local Search

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  - Soft restart: Apply a modification to p_{cur} which the current search moves cannot do, i.e., move outside of the current neighborhood of p_{cur} without throwing it away completely.

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Iterated Local Search

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- It can also be applied to Simulated Annealing, or to Hill Climbers if, e.g., they do not find improvements for several steps.
- Such searches are called *Iterated Local Search* (ILS) [4]
- They have astonishingly great performance, and several of the best application-specific optimization methods are based on them [4, 23].

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1. Introduction
2. Tabu Search
3. Example 1: MAX-SAT
4. Example 2: Traveling Salesman Problem
5. Iterated Local Search
6. Summary

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- Tabu Search, Simulated Annealing, and many other local search algorithms can be *iterated* by making stronger search moves or restarting them altogether from time to time.

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- It is based on the concept of scanning complete neighborhoods and avoiding to move in a cycle by declaring certain solution features as “tabu”.
- The concept of search moves here is slightly different from the algorithms we discussed before and will discuss afterwards, it is centered around neighborhoods rather than single modifications.
- Tabu Search, Simulated Annealing, and many other local search algorithms can be *iterated* by making stronger search moves or restarting them altogether from time to time.
- We have also looked into two very well-known, classical problems from operations research again, Maximum Satisfiability and the Traveling Salesman Problem.

谢谢

Thank you

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Caspar David Friedrich, “Der Wanderer über dem Nebelmeer”, 1818
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