Blending Dynamic Programming with Monte Carlo Simulation for Bounding the Running Time of Evolutionary Algorithms

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Introduction

- **Dynamic parameter settings** can greatly improve the efficiency of evolutionary algorithms (EAs)
- **Runtime lower bounds** give a baseline, which is important for algorithm comparison and development
- Proving precise lower bounds for algorithms with dynamic parameter choices is challenging
- Previously, a dynamic programming approach was proposed to derive lower bounds for simple problems [Buzdalov, Doerr, PPSN 2020]
  - transition probabilities between different states can be expressed by mathematical expressions
  - applied to derive optimal mutation rates for OneMax problem
- We propose a method that combines dynamic programming with Monte Carlo sampling, which is applicable for a broader problem class
Considered Evolutionary Algorithms

**Data:** $n$: problem size; $f : \{0, 1\}^n \rightarrow \mathbb{R}$: function to maximize; $\lambda$: population size; $\mathcal{D}(p)$: a family of parameterized distributions over $[0..n]$

1. Sample parent $x \in \{0, 1\}^n$ uniformly at random;
2. For $t \leftarrow 1, 2, \ldots$ do
   3. For $i \in [1..\lambda]$ do
      4. Choose a distribution parameter $p^t_i$;
      5. Sample $k_i \sim \mathcal{D}(p^t_i)$, the number of bits to flip;
      6. Create $y_i$ by flipping $k_i$ different bits in $x$ chosen uniformly at random;
   7. Select $x \leftarrow \arg\max_{z \in \{x, y_1, \ldots, y_{\lambda}\}} f(z)$ breaking ties arbitrarily;
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Parameter control in $(1 + \lambda)$ EA with mutation rate $p$:

- 2-rate: try $p/2$ and $2p$ on two halves of population
- Ab rule: multiply $p$ by $A$ or $b$ based on success
- HQEA: multiply $p$ by $A$ or $b$ according to Q-learning
Ruggedness Problem and Benchmarking

Optimum: \( f(z) = n \).
Points at Hamming distance one from \( z \) have fitness \( n - 2 \),
those at distance two have fitness \( n - 1 \),
those at distance three have fitness \( n - 4 \),
those at distance four have fitness \( n - 3 \), and so on.

Previous results for parameter control on Ruggedness:

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
  i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  r_2(i) & 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 & 9 & 8 & 10 \\
\end{array}
\]
Description of the Proposed Method

1. $f_{\min}, f_{\max} \leftarrow \text{minimum and maximum fitness values};$
2. Initialize optimal times: $T_{f_{\max}}^* \leftarrow 0;$
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      5. Compute approximate probabilities $(\tilde{p}_i)_{i=0,1,\ldots}$ of increasing fitness by $i$ with mutation rate $p$ using the Monte Carlo approach;
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      6. \( T_{f,p} \leftarrow \frac{1}{1 - \tilde{p}_0} \left( 1 + \sum_{i>0} \tilde{p}_i \cdot T_{f+i}^* \right); \)
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6. Store optimal time: $T^*_f \leftarrow \min_p(T_{f,p})$;
7. Store optimal rate: $P_{f_{\text{opt}}} \leftarrow \arg\min_p(T_{f,p})$;
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7. Store optimal time: $T_f^* \leftarrow \min_p(T_{f,p});$
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9. return $\{P_{f}^{\text{opt}}, T^*, T\}$
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Requirement: the optimal choice of \( p \) depends on the fitness value exclusively
Lower Runtime Bounds for Parameter Control

Iterations until the optimum of OneMax (left) and Ruggedness (right)

- New insight: on Ruggedness, only a constant-factor improvement is possible
- Why does (A,b) rule performs so much worse than 2-rate when using $p_{\text{min}} = 1/n^2$?
Optimal Mutation Rates

- Regular oscillations on Ruggedness with a period of 2
- It may be difficult to track precisely – is this a problem?
Parameter Efficiency Heatmaps

- Relative efficiency of the corr. $p$ among all mutation rates for the corr. $f$
- The range of nearly equally good rates is wide enough
- On Ruggedness, for odd fitness values the best rates are higher
- (A, b) rule (red) gets stuck with too small rates near the optimum
Regret Plots

Regret $|T_{f,p} - T_f^*|$ for $p$ chosen by 2-rate (left) and $(A, b)$ rule (right)

- How much of the performance the method loses from acting suboptimally
- $(A, b)$ rule spends most of its time with very large regrets
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  - Useful for deriving optimal rates and runtime lower bounds
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- Example application
  - Runtime estimations for the $(1 + \lambda)$ EA on the Ruggedness problem, $n = 100$
  - Analysis of $(A, b)$ and 2-rate parameter control methods
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» Possible solution:
  » construction of Markov chains on all states with equal fitness
  » solving the resulting system of equations
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Thank you!