# Novel Zigzag-based Benchmark Functions for Bound Constrained Single Objective Optimization

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- we propose novel zigzag-based benchmark functions for bound constrained single objective optimization:
  - non-differentiable
  - highly multimodal
  - with “tunable” difficulty
- we report on computational experiments where we compare two state-of-the-art algorithms and one standard EA on a set of problems that utilize the new benchmark functions

# Novel Benchmark Functions

The new benchmark functions are constructed as follows. First, we devise a "zigzag" function $z(x)$. For given parameters $k > 0, m > 0$ the zigzag function $z(x)$ at a point $x \in \mathbb{R}$ is computed as:

$$z(x) = \begin{cases} m\Big(\frac{1}{2} + (-1)^{\lceil kx \rceil}\big(\frac{\lceil kx \rceil + \lfloor kx \rfloor}{2} - kx\big)\Big), \text{if } (kx) \notin \mathcal{Z} \\ 0, \quad \text{if } \frac{kx}{2} \in \mathcal{Z} \\ m, \quad \text{otherwise,} \end{cases}$$

where $\frac{2}{k}$ is the period and $m$ is the amplitude of the zigzag function, as depicted in following figure.

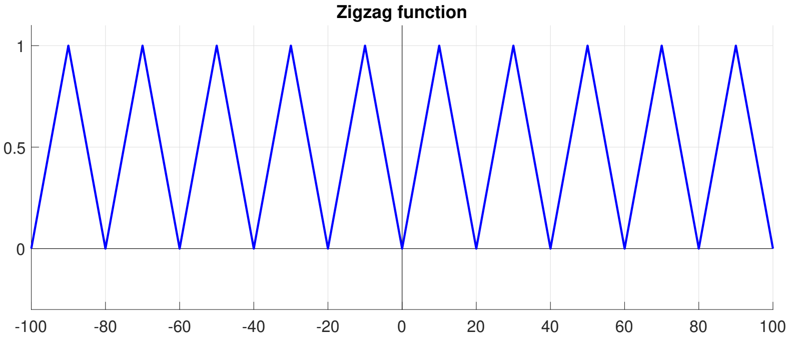



Figure: Zigzag function $z(x)$ with $k = 0.1, m = 1$.

The next step is a construction of a multimodal function $f(x)$:

- a sum of an absolute value of a high degree polynomial with one root in zero and an absolute value function
- the scaling of these two parts is such that the function values on the interval $[-200, 200]$ lie between $[0, 200]$ (allows composing the function with itself without severe numerical difficulties)
- the "polynomial" part of the function $f$ is then multiplied with the zigzag function $z(x)$
- the particular choice for the function $f(x)$ in this paper is the following:

$$f(x){=}3{\cdot}10^{-9}|(x{-}50)(x{-}190)x(x{+}70)(x{+}180)|z(x) + 0.2|x|$$

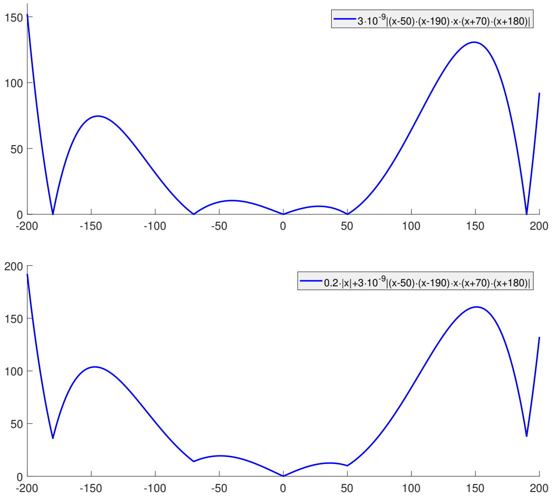
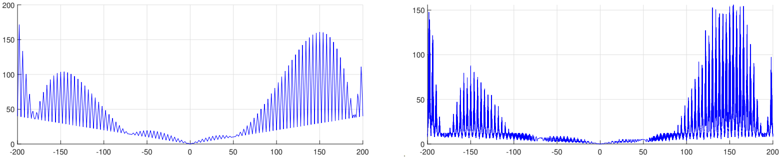
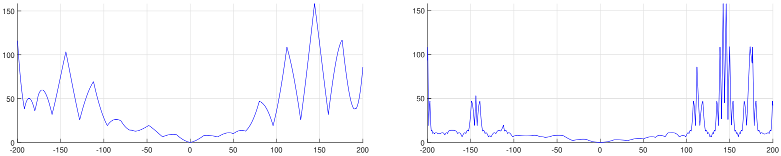



Figure: Partial construction of the function $f(x)$.




$f(x)$ and $f(f(x))$ for $k = 2^{-1}$




$f(x)$ and $f(f(x))$ for $k = 2^{-4}$

Figure: Functions $f(x)$ (left) and $f(f(x))$ (right) for different values of parameter $k$

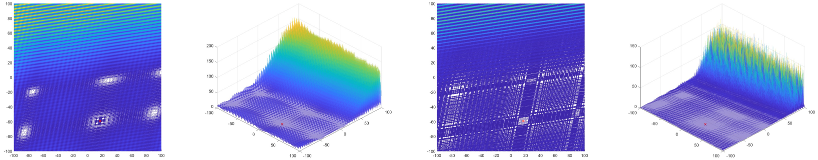
- the function $f$ has a single global optimum point in 0, is non-differentiable and highly multimodal (the "degree of multimodality" depending on the parameter $k$)
- finally, the two proposed benchmark functions $F_1(\mathbf{x})$ and $F_2(\mathbf{x})$, for $\mathbf{x} = [x_1, \ldots, x_D]^T$ and $\mathbf{x} \in [-100, 100]^D$, are the following:

$$f_1(\mathbf{x}) = \sum_{i=1}^{D} f(x_i), \qquad F_1(\mathbf{x}) = f_1(\mathbf{M_1}(\mathbf{x} - \mathbf{s_1})),$$

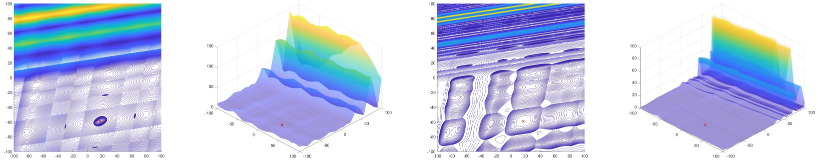
$$f_2(\mathbf{x}) = \sum_{i=1}^{D} f(f(x_i)), \qquad F_2(\mathbf{x}) = f_2(\mathbf{M_2}(\mathbf{x} - \mathbf{s_2})),$$

  where $\mathbf{s_1}, \mathbf{s_2} \in [-100, 100]^D$ are random shifts of the optimal solution and $\mathbf{M_1}, \mathbf{M_2}$ are random rotation/scaling matrices, with eigenvalues in the range $[0.5, 1]$

values can be found in: https://github.com/JakubKudela89/Zigzag



$F_1(\mathbf{x})$ and $F_2(\mathbf{x})$ for $k = 2^{-1}$



$F_1(\mathbf{x})$ and $F_2(\mathbf{x})$ for $k = 2^{-4}$

Figure: Contour and surface plots of the benchmark functions $F_1(\mathbf{x})$ (left) and $F_2(\mathbf{x})$ (right) for different values of parameter $k$. The optimum is highlighted by a red marker.

# Computational Experiments and Results

For computational experiments with the two benchmark functions we chose three algorithms:

- the first algorithm is the canonical particle swarm optimization (PSO) algorithm
- the second algorithm is the winner of the CEC'20 Competition on Single Objective Bound Constrained Optimization, the Improved Multi-operator Differential Evolution (IMODE) algorithm
- the third algorithm is the runner-up of the same competition, the Adaptive Gaining-Sharing Knowledge (AGSK) based algorithm

We use the same benchmark rules as the CEC'20 competition:

- the three algorithms are evaluated on the two benchmark functions with $D = [5, 10, 15, 20]$ dimensions
- parameter $k = [2^0, 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}]$
- search space of $[-100, 100]^D$
- maximum number of function evaluations were set to 50,000, 1,000,000, 3,000,000 and 10,000,000 fitness function evaluations for problems with $D = [5, 10, 15, 20]$
- all algorithms were run 30 times to obtain representative results
- for both IMODE and AGSK, we use the same parameter settings that they used in the CEC'20 competition

## Results for $F_1(x)$ :

| | | D = 5 | | | D = 10 | | | D = 15 | | | D = 20 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | PSO | AGSK | IMODE | PSO | AGSK | IMODE | PSO | AGSK | IMODE | PSO | AGSK | IMODE |
| $k = 2^0$ | min | **0** | **0** | 9.37e–06 | 1.73e–04 | **0** | 1.12e+00 | **2.80e–01** | 2.53e+00 | 2.70e+00 | **1.01e+00** | 5.23e+00 | 6.06e+00 |
| | median | **0** | **0** | 1.76e–02 | **1.64e–01** | 5.22e–01 | 1.91e+00 | **1.91e+00** | 5.06e+00 | 3.90e+00 | **2.87e+00** | 7.67e+00 | 8.44e+00 |
| | mean | **0** | 1.87e–02 | 6.98e–02 | **4.08e–01** | 7.61e–01 | 1.93e+00 | **1.92e+00** | 4.89e+00 | 3.86e+00 | **3.18e+00** | 7.67e+00 | 8.51e+00 |
| | max | **0** | 2.77e–01 | 4.08e–01 | 2.42e+00 | **2.37e+00** | 2.73e+00 | **4.33e+00** | 6.31e+00 | 5.24e+00 | **6.94e+00** | 9.72e+00 | 1.03e+01 |
| | std | 0 | 5.47e–02 | 7.46e–01 | 6.57e–01 | 7.48e–01 | 4.71e–01 | 1.12e+00 | 9.45e–01 | 5.92e–01 | 1.60e+00 | 1.15e+00 | 9.07e–01 |
| $k = 2^{-1}$ | min | **0** | **0** | 6.75e–08 | 9.22e–06 | **0** | 3.68e–01 | 8.81e–01 | **0** | 2.64e+00 | **1.12e+00** | 5.52e+00 | 7.27e+00 |
| | median | **0** | **0** | 8.84e–04 | **4.45e–01** | 9.04e–01 | 2.34e+00 | **3.38e+00** | 4.90e+00 | 4.02e+00 | **3.39e+00** | 8.76e+00 | 8.85e+00 |
| | mean | **0** | 6.89e–02 | 4.23e–02 | **6.80e–01** | 1.02e+00 | 2.25e+00 | **3.49e+00** | 4.80e+00 | 3.99e+00 | **3.60e+00** | 8.67e+00 | 8.88e+00 |
| | max | **2.57e–08** | 1.00e+00 | 2.67e–01 | **1.91e+00** | 2.74e+00 | 3.19e+00 | 6.58e+00 | 6.11e+00 | **5.10e+00** | **7.61e+00** | 1.03e+01 | 1.05e+01 |
| | std | 0 | 2.00e–01 | 7.98e–02 | 6.34e–01 | 7.84e–01 | 6.64e–01 | 1.64e+00 | 1.14e+00 | 5.72e–01 | 1.38e+00 | 1.13e+00 | 7.54e–01 |
| $k = 2^{-2}$ | min | **0** | **0** | **0** | 2.14e–03 | **0** | 1.31e+00 | 9.60e–01 | 3.01e+00 | **5.40e–01** | **8.02e–01** | 6.65e+00 | 4.25e+00 |
| | median | **0** | 1.10e–07 | 1.34e–04 | **6.44e–01** | 1.67e+00 | 2.13e+00 | 3.47e+00 | 5.01e+00 | **3.16e+00** | **3.37e+00** | 8.49e+00 | 8.08e+00 |
| | mean | 3.73e–02 | 3.25e–02 | **2.99e–02** | **6.24e–01** | 1.59e+00 | 2.12e+00 | 3.70e+00 | 4.82e+00 | **3.04e+00** | **3.53e+00** | 8.48e+00 | 7.91e+00 |
| | max | 8.02e–01 | **2.87e–01** | 5.22e–01 | **1.61e+00** | 2.99e+00 | 3.65e+00 | 7.54e+00 | 5.72e+00 | **4.06e+00** | **7.86e+00** | 1.01e+01 | 9.61e+00 |
| | std | 1.55e–01 | 7.68e–02 | 1.13e–01 | 3.26e–01 | 7.84e–01 | 5.39e–01 | 1.86e+00 | 7.13e–01 | 7.79e–01 | 1.30e+00 | 9.56e–01 | 1.09e+00 |
| $k = 2^{-3}$ | min | **0** | **0** | **0** | 1.48e–05 | **0** | 8.41e–04 | **0** | 2.94e+00 | 8.67e–04 | **0** | 5.96e+00 | 2.74e+00 |
| | median | **0** | **0** | **0** | **3.79e–02** | 3.20e–01 | 1.55e+00 | 3.85e+00 | 4.65e+00 | **9.60e–01** | **3.70e+00** | 7.91e+00 | 5.83e+00 |
| | mean | **0** | 2.79e–04 | 1.52e–06 | **4.21e–01** | 8.00e–01 | 1.41e+00 | 3.75e+00 | 4.58e+00 | **8.23e–01** | **3.64e+00** | 8.09e+00 | 5.78e+00 |
| | max | **0** | 8.14e–03 | 2.22e–05 | **1.92e+00** | 2.74e+00 | 2.36e+00 | 8.06e+00 | 5.71e+00 | **1.65e+00** | **7.17e+00** | 1.05e+01 | 7.53e+00 |
| | std | 0 | 1.48e–03 | 5.48e–06 | 5.51e–01 | 9.79e–01 | 6.59e–01 | 1.92e+00 | 6.86e–01 | 5.20e–01 | 1.71e+00 | 1.04e+00 | 1.28e+00 |
| $k = 2^{-4}$ | min | **0** | **0** | **0** | 3.97e–06 | **0** | **0** | **0** | **0** | **0** | **0** | 3.83e+00 | 1.28e+00 |
| | median | **0** | **0** | **0** | 4.59e–03 | **0** | **0** | 3.20e+00 | 3.83e+00 | **0** | **3.07e+00** | 7.60e+00 | 3.23e+00 |
| | mean | **0** | **0** | **0** | 2.00e–01 | 8.53e–02 | **3.49e–07** | 3.41e+00 | 3.62e+00 | **0** | 3.21e+00 | 7.37e+00 | **2.96e+00** |
| | max | **0** | **0** | **0** | 1.28e+00 | 1.28e+00 | **7.75e–06** | 8.88e+00 | 4.99e+00 | **0** | 6.12e+00 | 9.06e+00 | **3.97e+00** |
| | std | 0 | 0 | 0 | 4.11e–01 | 3.24e–01 | 1.48e–06 | 2.11e+00 | 1.19e+00 | 0 | 1.60e+00 | 1.17e+00 | 7.34e–01 |

## Results for $F_2(x)$ :

| | | D = 5 | | | D = 10 | | | D = 15 | | | D = 20 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | PSO | AGSK | IMODE | PSO | AGSK | IMODE | PSO | AGSK | IMODE | PSO | AGSK | IMODE |
| $k = 2^0$ | min | **0** | **0** | 4.44e–05 | 1.70e–05 | **0** | 1.01e–01 | **9.66e–02** | 3.64e–01 | 2.40e–01 | **8.69e–02** | 6.03e–01 | 5.26e–01 |
| | median | **0** | 6.20e–03 | 1.20e–02 | **1.68e–02** | 1.69e–01 | 2.13e–01 | **3.65e–01** | 6.85e–01 | 4.15e–01 | **4.75e–01** | 9.09e–01 | 8.01e–01 |
| | mean | **1.75e–02** | 2.24e–02 | 1.84e–02 | **4.03e–02** | 1.61e–01 | 2.03e–01 | 4.91e–01 | 6.47e–01 | **4.00e–01** | **4.99e–01** | 8.89e–01 | 7.81e–01 |
| | max | 1.30e–01 | 1.25e–01 | **5.18e–02** | **1.92e–01** | 2.87e–01 | 2.61e–01 | 1.56e+00 | 8.40e–01 | **4.94e–01** | 2.23e+00 | 1.08e+00 | **9.30e–01** |
| | std | 2.95e–02 | 3.19e–02 | 1.72e–02 | 4.95e–02 | 7.54e–02 | 4.38e–02 | 4.34e–01 | 1.12e–01 | 6.23e–02 | 3.87e–01 | 1.24e–01 | 9.74e–02 |
| $k = 2^{-1}$ | min | **0** | **0** | 7.68e–06 | 3.68e–07 | **0** | 1.31e–01 | **1.11e–02** | 3.73e–01 | 2.90e–01 | **5.30e–02** | 4.25e–01 | 6.77e–01 |
| | median | **0** | 9.77e–08 | 1.91e–03 | **7.61e–03** | 1.22e–01 | 2.39e–01 | **3.16e–01** | 6.73e–01 | 5.40e–01 | **3.42e–01** | 9.89e–01 | 1.18e+00 |
| | mean | **0** | 3.74e–03 | 5.26e–03 | **2.32e–02** | 1.29e–01 | 2.52e–01 | **3.41e–01** | 6.44e–01 | 5.32e–01 | **3.81e–01** | 9.98e–01 | 1.16e+00 |
| | max | **0** | 3.67e–02 | 2.81e–02 | **1.23e–01** | 3.67e–01 | 4.12e–01 | 8.91e–01 | 8.59e–01 | **6.68e–01** | **9.25e–01** | 1.32e+00 | 1.37e+00 |
| | std | 0 | 9.20e–03 | 6.91e–03 | 3.68e–02 | 1.08e–01 | 7.34e–02 | 2.24e–01 | 1.21e–01 | 9.66e–02 | 2.33e–01 | 2.14e–01 | 1.39e–01 |
| $k = 2^{-2}$ | min | **0** | **0** | 1.32e–08 | 8.14e–06 | **0** | 4.56e–02 | **1.06e–01** | 1.38e–01 | 1.57e–01 | **2.05e–01** | 2.69e–01 | 5.38e–01 |
| | median | **0** | **0** | 1.17e–05 | **6.24e–03** | 1.29e–01 | 2.12e–01 | 3.68e–01 | 4.74e–01 | **3.11e–01** | **4.06e–01** | 9.12e–01 | 8.98e–01 |
| | mean | **1.83e–04** | 2.44e–03 | 1.99e–03 | **2.82e–02** | 1.27e–01 | 1.96e–01 | 4.00e–01 | 4.57e–01 | **3.20e–01** | **4.27e–01** | 8.84e–01 | 8.82e–01 |
| | max | **5.49e–03** | 2.35e–02 | 2.71e–02 | **2.00e–01** | 3.62e–01 | 3.38e–01 | 9.49e–01 | 6.54e–01 | **4.37e–01** | **7.04e–01** | 1.25e+00 | 1.10e+00 |
| | std | 1.00e–03 | 5.73e–03 | 5.61e–03 | 4.80e–02 | 8.63e–02 | 7.96e–02 | 2.18e–01 | 1.23e–01 | 6.62e–02 | 1.44e–01 | 1.91e–01 | 1.33e–01 |
| $k = 2^{-3}$ | min | **0** | **0** | **0** | 2.67e–08 | **0** | 4.74e–02 | **0** | 1.12e–01 | 1.45e–02 | **7.57e–02** | 2.93e–01 | 2.98e–01 |
| | median | **0** | **0** | 3.32e–08 | **1.38e–04** | 7.16e–02 | 8.30e–02 | 3.02e–01 | 2.60e–01 | **1.04e–01** | **2.53e–01** | 5.87e–01 | 3.93e–01 |
| | mean | 1.08e–03 | 1.04e–03 | **4.02e–06** | **2.84e–02** | 7.51e–02 | 9.08e–02 | 2.76e–01 | 2.54e–01 | **9.97e–02** | **2.77e–01** | 5.52e–01 | 3.96e–01 |
| | max | 3.26e–02 | 1.42e–02 | **4.90e–05** | **1.27e–01** | 1.59e–01 | 1.72e–01 | 5.15e–01 | 3.32e–01 | **1.63e–01** | **4.94e–01** | 7.02e–01 | 5.05e–01 |
| | std | 5.95e–03 | 3.35e–03 | 1.07e–05 | 4.18e–02 | 5.08e–02 | 3.00e–02 | 1.30e–01 | 5.24e–02 | 3.69e–02 | 1.21e–01 | 9.42e–02 | 4.94e–02 |
| $k = 2^{-4}$ | min | **0** | **0** | **0** | 6.19e–08 | **0** | **0** | **0** | **0** | **0** | 2.19e–02 | 1.12e–01 | **9.46e–05** |
| | median | **0** | **0** | **0** | 4.38e–05 | **0** | **0** | 8.40e–02 | 1.53e–01 | **0** | 1.97e–01 | 3.05e–01 | **1.54e–01** |
| | mean | **0** | **0** | **0** | 2.97e–03 | **0** | 1.00e–06 | 1.22e–01 | 1.47e–01 | **0** | 2.01e–01 | 2.91e–01 | **1.52e–01** |
| | max | **0** | **0** | **0** | 2.19e–02 | **0** | 1.12e–05 | 3.91e–01 | 2.28e–01 | **0** | 3.70e–01 | 3.85e–01 | **2.16e–01** |
| | std | 0 | 0 | 0 | 7.57e–03 | 0 | 2.68e–06 | 1.16e–01 | 4.86e–02 | 0 | 7.70e–02 | 6.51e–02 | 4.33e–02 |

- it is clear from the results that both the benchmark functions are not "impossible" to optimize, as there were plenty of instances where the algorithms found the optimal solution
- however, the instances are not "too easy" so that the algorithms find the optimum reliably – this, in our opinion, makes these benchmark functions worth investigating
- another observation to be made is that for both test functions, reducing the zigzag parameter $k$ really reduces the complexity of the problems. In particular, the changes from $k = 2^{-2}$ to $k = 2^{-3}$ and then to $k = 2^{-4}$ seem to have the biggest impact, while the statistical results for problems with $k = [2^0, 2^{-1}, 2^{-2}]$ are relatively stable
- unsurprisingly, the difficulty of the test problems also increases with the dimension $D$

- the most surprising results come from the comparison of the three algorithms
- overall, on both of the newly proposed benchmark functions, neither of the two best algorithms from the CEC'20 competition performed significantly better than a standard PSO
- we argue that this a prime reason for investigating these benchmark functions even further and for including them in future competitions and benchmark suits
- future research will encompass comparing a wider selection of algorithms, and developing multimodal benchmark functions using the presented technique

Thank you for your attention!