Revisiting the Price of Anarchy for Non-atomic Congestion Games

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Joint work with
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Outline

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1.1 Motivation: traffic congestion in large cities

The 2017 Annual Traffic Report[1] of AMap shows that more than 81% of cities in China suffered from traffic congestion. The heavy congestion caused serious economic loss, e.g., loss of Beijing caused by congestion accounts for 3.1%[2] of its annual GDP in 2017.

[1] http://report.amap.com

[2] http://huiyan.baidu.com/reports/2017Q4_niandu.html

1.1 Motivation: measures for alleviating congestion

Two measures in practice: congestion pricing and license plate lottary.

(a) Congestion pricing in Singapore

(b) Plate lottary in Beijing

1.1 Motivation: congestion pricing and plate lottary

Congestion pricing considers the selfish routing behavior of travelers as the main cause of congestion, and tolls streets so as to guide travelers using reasonable paths and thus reduce congestion.

Question 1: does selfish routing really cause congestion?

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Question 2: how much does travel demand contribute to congestion?

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($\star$) Our work devotes to Question 1 and focuses on the limit analysis of the price of anarchy (PoA) for non-atomic congestion games.

1.2 Non-atomic congestion games

A non-atomic congestion game (NCG) can be represented by a tuple

$$\Gamma = \left(\mathcal{K}, d = (d_k)_{k\in\mathcal{K}}, A, \mathcal{S} = \bigcup_{k\in\mathcal{K}} \mathcal{S}_k, r = \big(r(a,s)\big)_{a\in A, s\in\mathcal{S}}, \tau = (\tau_a)_{a\in A}\right),$$

where:

- $\mathcal{K} = \{1, \ldots, K\}$ is a finite set of user (player) groups, corresponding to the K O/D pairs in traffic.
- Each d_k is the *volume* of users in group $k \in \mathcal{K}$, corresponding to the *travel demand* of the k-th O/D pair.
- A is a finite set of *resources*, corresponding to the arc set of the road network.
- Each $\mathcal{S}_k$ is the set of *strategies* only available to group k, corresponding to the set of feasible paths for the k-th O/D.
- Each $r(a, s) \geq 0$ is a constant denoting the volume of resource $a \in A$ demanded by a user adopting strategy $s \in \mathcal{S}$.
- Each $\tau_a : [0, +\infty) \to [0, +\infty)$ is a *nondecreasing* and *continuous* price (or latency) function.

1.2 Non-atomic congestion games

NCGs are non-cooperative games of perfect information and popular static models for road traffic. In an NCG Γ, a feasible strategy profile is usually written as a vector $f = (f_s)_{s\in\mathcal{S}}$, where each $f_s \geq 0$ repesents the volume of users adopting strategy $s \in \mathcal{S}$ and $\sum_{s\in\mathcal{S}_k} f_s = d_k$ for all $k \in \mathcal{K}$.

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Consider an NCG Γ and a profile $f = (f_s)_{s\in\mathcal{S}}$, the cost of users adopting strategy $s \in \mathcal{S}$ equals

$$\tau_s(f) := \sum_{a\in A} r(a,s)\tau_a(f_a),$$

where $f_a = \sum_{s\in\mathcal{S}} r(a,s)\cdot f_s$ is the consumed volume of resource $a \in A$ under profile f.

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The social cost of f then equals

$$C(f) := \sum_{a\in A} r(a,s)\cdot\tau_a(f_a) = \sum_{k\in\mathcal{K}}\sum_{s\in\mathcal{S}_k} f_s\cdot\tau_s(f).$$

1.3 Nash equilibrium and the price of anarchy

The selfish behavior of users may lead the underlying game into a *steady state*, called Nash equilibrium (NE). A feasible profile $\tilde{f} = (\tilde{f}_s)_{s\in\mathcal{S}}$ is at NE if

$$\forall k \in \mathcal{K} \forall s, s' \in \mathcal{S}_k \Big(\tilde{f}_s > 0 \implies \forall \epsilon \in (0, \tilde{f}_s) \big(\tau_s(\tilde{f}) \leq \tau_{s'}(f^{1+\epsilon}) \big) \Big),$$

where

$$f^{1+\epsilon}_{s''} = \begin{cases} \tilde{f}_s - \epsilon, & \text{if } s'' = s, \\ \tilde{f}_{s'} + \epsilon, & \text{if } s'' = s', \\ \tilde{f}_{s''}, & \text{otherwise.} \end{cases}$$

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In an NE, users will not tend to change their strategies, since no unilateral change in strategy can introduce extra profit!

1.3 Nash equilibrium and the price of anarchy

An NE profile approximates the decision of users in practice. A feasible profile minimizing the social cost is called system optimum (SO). Thus the *worst ratio of the social cost of NE profiles over that of SO profiles* could be considered to be an measurement of the inefficiency of the user (selfish) behavior.

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Definition 1 (Papadimitriou 1999 LNCS) Consider an NCG

$$\Gamma = (\mathcal{K}, d, A, \mathcal{S}, r, \tau)$$

The price of anarchy (PoA) of Γ w.r.t. user volume vector $d = (d_k)_{k\in\mathcal{K}}$ equals

$$\text{PoA}(d) := \sup\left\{\frac{C(\tilde{f})}{C(f^*)} : \tilde{f} \text{ is an NE profile}\right\},$$

where f^* is an arbitrary SO profile.

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In our model, the NE must exist and all NE profiles will have the same cost, see [Smith 1979 TPB]. Thus,

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To check whether selfish routing significantly contribute to congestion, one needs to closely analyze the PoA for heavy traffic, i.e., the case that total volume $T(d) := \sum_{k \in \mathbb{N}} d_k$ is large.

1.4 Basic properties of NE and SO profiles

Lemma (Roughgarden & Tardo 2002 JACM)

1) $\tilde{f}$ is an NE profile if and only if for each $k \in \mathcal{K}$ and $s, s' \in \mathcal{S}_k$ with $\tilde{f}_s > 0$, $\tau_s(\tilde{f}) \leq \tau_{s'}(\tilde{f})$.
2) If all $\tau_a(\cdot)$ are differentiable, then f^* is an SO profile if and only if for each $k \in \mathcal{K}$ and $s, s' \in \mathcal{S}_k$ with $f^*_s > 0$, $\sum_{a \in A} r(a,s)\big(f^*_a \cdot \tau'_a(f^*_a) + \tau_a(f^*_a)\big) \leq \sum_{a \in A} r(a,s')\big(f^*_a \cdot \tau'_a(f^*_a) + \tau_a(f^*_a)\big)$.

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Remark:

An SO profile is an NE profile w.r.t. "price" functions $x \cdot \tau_a'(x) + \tau_a(x)$.

1.4 Basic properties of NE and SO profiles

In general, it is difficult to compute NE profiles. However, when all $\tau_a(\cdot)$ are continuous and nondecreasing, we can compute NE by solving the following convex program.

$$\min \quad \sum_{a\in A} \int_0^{x_a} \tau_a(t)dt$$

$$s.t.$$

$$\sum_{s\in\mathcal{S}_k} x_s = d_k, \forall k \in \mathcal{K},$$

$$x_s \geq 0, \forall s \in \mathcal{S}_k, \forall k \in \mathcal{K}.$$

This program can be well solved by a revised Frank-Wolfe algorithm [John et al. 2005 OR].

1.4 Basic properties of NE and SO profiles

Obviously, we can compute SO by solving the following program.

$$
\begin{aligned}
\min \quad & \sum_{a\in A} x_a \cdot \tau_a(x_a) \\
s.t. \quad & \\
& \sum_{s\in\mathcal{S}_k} x_s = d_k, \forall k \in \mathcal{K}, \\
& x_s \geq 0, \forall s \in \mathcal{S}_k, \forall k \in \mathcal{K}.
\end{aligned}
$$

The above program can also be well solved by the Frank-Wolfe algorithm when all $\tau_a(\cdot)$ are convex and differentiable.

1.4 Basic properties of NE and SO profiles

Frank-Wolfe algorithm is an iterative search. Consider a program

$$\min \quad h(x) \qquad s.t.\ x \in P$$

for an objective function $h(\cdot)$ and $P \subseteq \mathbb{R}^n$ denoting the feasible region for some $n \in \mathbb{N}_+$. The algorithm starts with a fixed initial point $x^{(0)} \in P$, and then iteratively solves the linear approximation of the program so as to anew the current point $x^{(t)}$. In details, the iteration t of the algorithm consists of the following two steps:

S1) $y = \arg\min_{x\in P} \nabla h(x^{(t)})^T \cdot x$, where $\nabla h(x^{(t)})$ is the gradient of $h(\cdot)$ at point $x^{(t)}$.

S2) $\theta^* = \arg\min_{\theta\in[0,1]} h(\theta \cdot x^{(t)} + (1-\theta)\cdot y)$ and put $x^{(t+1)} = \theta^* \cdot x^{(t)} + (1-\theta^*)\cdot y$.

Step $S2)$ is a line search that can be done by binary search.

1.4 Basic properties of NE and SO profiles

For the computation of NE and SO profiles, step S1) is actually equivalent to a linear program that can be solved by Simplex.

In particular, if $r(a,s)=0$ or 1 for all $a\in A$ and $s\in S$, then S1) is a shortest path problem that can be efficiently solved by Dijkstra's algorithm. For the computation of NE, the "arc length" of $a\in A$ is just $\tau_a(x_a^{(t)})$. And the "arc length" of $a\in A$ equals $x_a^{(t)}\cdot\tau_a'(x_a^{(t)})+\tau_a(x_a^{(t)})$ for the computation of SO profiles. Herein, we recall that $x^{(t)}=(x_s^{(t)})_{s\in\mathcal{S}}$ is a strategy profile for each step t.

There are open source software available for the computation of NE profiles and SO profiles. For instance, the MatSim (by Kai Nagel from TU Berlin) and CMCF (by the COGA group led by Martin Skutella at TU Berlin).

Although there are well designed softwares, it is still of challenge to compute the PoA in practice for games with a large number of groups, due to the prohibitive computational time.

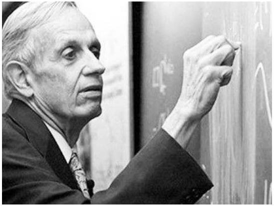
Outline

2.1 Popular subfields of game theory

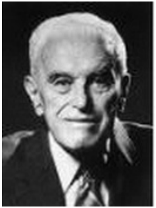
Equilibria analysis (since 1950):

The existence of equilibria, the properties of equilibira, and others, e.g., [Nash 1950 Dissertation], [Wordrop 1952 PICE], [Smith 1979 TPB], [Harsanyi 1982 Book], [Selten 1988 Book] and others.

($\star$) 1994 Nobel Prize in Economics: Nash, Harsanyi & Selten (for their notable works on the equilibrium analysis of noncooperative games)



(a) J.F.Nash Jr.



(b) J.C. Harsanyi

(c) R. Selten

Figure: Winners of the 1994 Nobel Prize in Economics

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Mechnism design (since 1999):

Design game-theoretic mechanism for particular purpose, for instance for auctions, see, e.g., [Noam & Ronen 1999 STOC], [Noam & Ronen 2001 GEB], [Paul & Andreas 2007 Report], [Vazirani et al. 2007 Book] and others.

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Worst-case analysis of the PoA (since 1999):

Analyze the upper bound of the PoA for NCGs and other games, see, e.g., [Papadimitriou 1999 LNCS], [Roughgarden & Tardos 2002 JACM], [Koutsoupias & Papadimitriou 2009 CSR] and others.
(⋆) 2012 Gödel Prize: Roughgarden, Tardos, Koutsoupias & Papadimitriou (for the notable works [Roughgarden & Tardos 2002 JACM] and [Koutsoupias & Papadimitriou 2009 CSR] on worst-case analysis of PoA).

2.2 Worst-case upper bound is not sufficient

[Roughgarden 2001 FOCS], [Roughgarden & Tardos 2002 JACM], [Roughgarden & Tardos 2002 GEB], [Roughgarden 2003 JCSS], [Roughgarden & Tardos 2007 Book] and others investigated the worst-case upper bound of the PoA for NCGs with different types of price functions $\tau_a(\cdot)$. Among others, the most famous results are:

Theorem (Roughgarden & Tardos 2002 JACM) If all price functions $\tau_a(\cdot)$ of an NCG Γ are affine linear, then the PoA$(d) \leq 4/3$ for all volume vector $d = (d_k)_{k\in\mathcal{K}}$. Moreover, there exists an NCG Γ with affine linear price functions s.t. the PoA$(d) = 4/3$ for some d. (was awarded the Gödel Prize in 2012)

Theorem (Roughgarden 2003 JCSS) Consider an NCG Γ. If all $\tau_a(\cdot)$ are polynomials with degree at most $\rho > 0$, then the PoA$(d) \in \Theta(\rho/\ln\rho)$ for all d.

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Worst-case upper bound analysis of the PoA pessimistically assumes the inefficiency of user behavior, and aims to bound such inefficiency. This is true to a certain extent, but is not the whole story.

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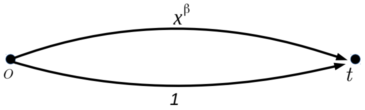

Figure: Pigou's game

Consider the Pigou's game [Pigou 1920 book]. The PoA equals $T/(T-(\beta+1)^{-1/\beta}+(\beta+1)^{-1-1/\beta})$, where T is the total volume. Considering all possible β, the PoA could be arbitrarily large when total volume $T=1$ is fixed.

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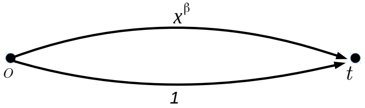

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2.2 Worst-case upper bound is not sufficient

Therefore, to impartially and comprehensively understand the user behavior in congestion games, one still needs a closer inspection of the PoA(d) for the case of that the total volume $T(d) = \sum_{k\in\mathcal{K}} d_k$ is large.

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This actually requires a limit analysis of the PoA(d) when $T(d)$ approaches infinity.

This kind of analysis is more reasonable, since the total volume of users in practice is often very large, for instance the total travel demand in Beijing.

2.3 Results in the last two years

In the last two years, [Colini et al. 2016 SAGT], [Colini et al. 2017 WINE] and [Wu et al. 2017 arXiv] initiated the limit analysis of the PoA. However, they only obtained "partial" convergence results.

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Theorem (Conili et al. 2016, 2017) Consider an NCG Γ with polynomial price functions. Then, $\lim_{n\to\infty} \mathrm{PoA}(d^{(n)}) = 1$ for each user volume vector sequence $\{d^{(n)}\}_{n\in\mathbb{N}}$ s.t. $\lim_{n\to\infty} T(d^{(n)}) = \infty$ and $\underline{\lim}_{n\to\infty} d_k^{(n)}/T(d^{(n)}) > 0$ for each $k \in \mathcal{K}$, where each $d^{(n)} = (d_k^{(n)})_{k\in\mathcal{K}}$ and $T(d^{(n)}) = \sum_{n\in\mathbb{N}} d_k^{(n)}$.

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Remark: This result requires particular structure of the user volume vectors when the total volume approaches ∞. This is obviously not the general case.

2.3 Results in the last two years

[Wu et al. 2017 arXiv] considered the convergence of $\mathrm{PoA}(d^{(n)})$ for arbitrary user volume vector sequence $\{d^{(n)}\}_{n\in\mathbb{N}}$. They proposed the concept of limit games and scalable games, and then obtained the following results.

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Theorem (Wu et al. 2017) Consider an NCG. If the game is scalable, then PoA(d) converges to 1 as $T(d)$ approaches infinity. Moreover, NCGs with polynomial price functions of the same degree ρ is scalable and PoA(d)$=1+O(T(d)^{-\rho})$.

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Theorem (Wu et al. 2017) Consider an NCG. If the game is scalable, then PoA(d) converges to 1 as $T(d)$ approaches infinity. Moreover, NCGs with polynomial price functions of the same degree ρ is scalable and PoA(d)$=1+O(T(d)^{-\rho})$.

Remark: This is actually a surprising result! Recall that [Roughgarden & Tardos 2002 JACM] proved that the worst-case upper bound of NCGs with affine linear price functions is $4/3$ and [Roughgarden & Tardos 2003 JCSS] proved that the worst-case upper bound is $\Theta(\rho/\ln\rho)$ if price functions are of degree ρ. However, they still failed to show the convergence for arbitrary polynomial functions.

Outline

3.1 Our contribution

We further formulated the idea of limit games stemming from [Wu et al. 2017 arXiv], and developed a general technique, called asymptotic decomposition, which allows us to thoroughly analyze the limit of PoA for NCGs with price functions of a large variety. In particular, we completely proved that the PoA of NCGs with polynomial price functions converges to 1 as the total volume of users approaches infinity. This result complements the notable results [Roughgarden & Tardos 2002 JACM] and [Roughgarden & Tardos 2003 JCSS] about the worst-case upper bound of PoA. Thus the user behavior might be inefficient when total volume T is not large. However, when T is large, the user behavior becomes efficient. This may because of the "swarm intelligence" hiding in the nature of human beings!!

3.2 Asymptotically well designed games and limit games

Definition (Wu et al. 17) An NCG Γ is called a *well designed* game (WDG) if the PoA$(d) \equiv 1$ for each user volume vector $d = (d_k)_{k \in \mathcal{K}}$ s.t. $T(d) = \sum_{k \in \mathcal{K}} d_k > 0$.

Definition (Wu et al. 17) An NCG Γ is called an *asymptotically well designed* game (AWDG) if PoA$(d) \to 1$ as $T(d) \to \infty$.

Remark:

In a WDG, the selfish behavior of users is efficient. These games thus are strongly favored in practice. [Wu et al. 2017 arXiv] shows examples for WDGs. However, in general, it is difficult to design WDGs in practice, see [Roughgarden 2001 FOCS]. NCGs intrinsically consider how to effectively allocate resources to a large number of users. Hence, a reasonable alternative to design an WDG is to design an AWDG. Most importantly, to our purpose, we need to explore properties of AWDG.

3.2 Asymptotically well designed games and limit games

The first exploration of AWDGs was done by [Colini et al. 2017 WINE], although the concept of AWDGs was first proposed by [Wu et al. 2017 arXiv]. [Colini et al. 2017 WINE] assumed a regularly varying function $g(\cdot)$ s.t. for all $a \in A$

$$\lim_{x\to\infty} \frac{\tau_a(x)}{g(x)} = q_a \in [0, \infty]$$

exists. With this $g(\cdot)$, they then classified strategies. Using the technique of *regular variation* [Bringham 1987 Book], they was able to prove the convergence of PoA for particular sequences $\{d^{(n)}\}_{n\in\mathbb{N}}$ of user volume vectors that are fully determined by $g(\cdot)$.

In their study, the function $g(\cdot)$ is not flexible and completely determined by the price functions $\tau_a(\cdot)$. Therefore, their technique can only apply to very special sequences $\{d^{(n)}\}_{n\in\mathbb{N}}$.

3.2 Asymptotically well designed games and limit games

[Wu et al. 2017 arXiv] observed the limitation of [Conili et al. 2017 WINE]. To avoid similar limitations in their study, they proposed the concept of scalable games and limit games.

3.2 Asymptotically well designed games and limit games

Definition 2 Consider an NCG Γ and a user volume vector sequence $\{d^{(n)}\}_{n\in\mathbb{N}}$ s.t. $T(d^{(n)}) \to \infty$ as $n \to \infty$. An NCG

$$\Gamma^{(\infty)} = \left(\mathcal{K}^{\infty}, \mathbf{d} = (\mathbf{d}_k)_{k\in\mathcal{K}^{\infty}}, \mathcal{S}^{\infty}, r, (\tau_a^{\infty})_{a\in A}\right)$$

is called a *limit* of Γ w.r.t. sequence $\{d^{(n)}\}_{n\in\mathbb{N}}$ if there exist an *infinite subsequence* $\{n_i\}_{i\in\mathbb{N}}$ and a sequence $\{g_i\}_{i\in\mathbb{N}}$ of *scaling factors* s.t.:

- Each $\tau_a^{\infty}(x) = \lim_{i\to\infty} \tau_a(T(d^{(n_i)})x)/g_i$ is either continuous and nondecreasing, or $\equiv \infty$ for all $x > 0$.
- Each group $k \in \mathcal{K}$ is either *negligible*, or contains a *tight strategy*. $\mathcal{K}^{\infty} \subseteq \mathcal{K}$ is the set of non-negligible groups and $\mathcal{S}^{\infty}$ is the set of all tight strategies.
- Each $\mathbf{d}_k = \lim_{i\to\infty} d_k^{(n_i)}/T(d^{(n_i)})$.
- Cost of NE profiles of Γ^{∞} w.r.t. volume vector $\mathbf{d}$ is positive.

3.2 Asymptotically well designed games and limit games

In Definition 2, a group k is called negligible if for each sequence $\{f^{(n_i)}\}_{i\in\mathbb{N}}$ of feasible strategy profiles,

$$\sum_{s\in\mathcal{S}_k} f_s^{(n_i)}\cdot\tau_s(f^{(n_i)})\in o(g_i),$$

and strategy $s\in\mathcal{S}$ is called tight if the limit price

$$\tau_a^{\infty}(\cdot)\not\equiv+\infty$$

for all $a\in A$ with $r(a,s)>0$.

Definition (Wu et al. 2017) An NCG Γ is called *scalable* if every subsequence of each sequence $\{d^{(n)}\}_{n\in\mathbb{N}}$ has a limit game Γ^{∞} that is well designed.

3.2 Asymptotically well designed games and limit games

The techniques used in the study of [Wu et al. 2017 arXiv] are basic calculus and thus very elementary. Although the tool is simple, their results cover those in [Colini et al. 2017 WINE]. In addition, they also showed by examples that their results are much more general than [Colini et al. 2017 WINE].

However, the technique of [Wu et al. 2017 arXiv] applies only to case that all groups share the same resources i.e. $\forall a \in A \forall k \in \mathcal{K} \exists s \in \mathcal{S}_k \big(r(a,s) > 0\big)$, or the resources demanded by different groups are mutually disjoint i.e. $\forall k, k' \in \mathcal{K}\Big(k \neq k' \to \forall a \in A \forall s \in \mathcal{S}_k \forall s' \in \mathcal{S}_{k'}\big(r(a,s)\cdot r(a,s') = 0\big)\Big)$.

3.2 Asymptotically well designed games and limit games

Therefore, to further explore AWDGs, the available techniques are far insufficient.

Through revisiting the technique of [Wu et al. 2017 arXiv], we developed a new technique, called asymptotic decomposition, and successfully applied to NCGs with polynomial (and more generally, regularly varying) price functions.

Our technique hybridizes these techniques used by [Wu et al. 2017 arXiv] and [Conili et al. 2017 WINE] as "subroutings". To generalize our technique, we additionally employed the idea of direct sum from algebra and considered the decomposition of games according to groups and their use of resources. The basic idea is to partition the groups into several subsets that are asymptotically independent of each other when their users determined strategies to follow.

3.3 Our results

Theorem 1 Consider an NCG Γ and a user volume vector sequence $\{d^{(n)}\}_{n\in\mathbb{N}}$ s.t. $T(d^{(n)}) \to \infty$ as $n \to \infty$. $\lim_{n\to\infty}\mathrm{PoA}(d^{(n)}) = 1$ if and only if the PoA of each limit Γ^{∞} of Γ w.r.t. $\{d^{(n)}\}_{n\in\mathbb{N}}$ equals 1.

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Theorem 2 Consider an NCG with polynomial price functions. Then, the PoA(d) converges to 1 as the total volume $T(d)$ of users approaches infinity.

Remark:
We actually also proved the same convergence for NCGs with regularly varying price functions. Regularly varying functions [Bingham1987Book] are very extensive, which includes most analytic functions that are popular in practice. Thus equilibrium in NCGs are in general need not be inefficient.

Outline

Asymptotic decomposition

General idea:

Our main result (Theorem 2) is obtained by *asymptotic decomposition*. The asymptotic decomposition is an iterative process. Consider an arbitrarily fixed $\{d^{(n)}\}_{n\in\mathbb{N}}$ s.t. $\lim_{n\to\infty} T(d^{(n)}) = \infty$. It aims to eventually partition the group set $\mathcal{K}$ into finitely many non-empty subsets $\mathcal{K}_0, \ldots, \mathcal{K}_t$ s.t. for each $m = 0, \ldots, t$, the marginal game

$$\Gamma_m := \Big(\bigcup_{u=0}^{m} \mathcal{K}_u, (d_k^{(n)})_{k\in\bigcup_{u=0}^{m}\mathcal{K}_u}, \mathcal{S}, r, \tau\Big)$$

has *well designed* limits (i.e., the PoA of the limit games always equal 1), and users from the marginal games are asymptotically independent of users from the remaining groups. If the decomposition can be done, then Theorem 1 will imply that the PoA$(d^{(n)})$ converges 1 as $n \to \infty$.

Asymptotic decomposition

Iterative step m:

Suppose that we have successfully constructed $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}$. We pick all the remaining groups $k \in \mathcal{K} \setminus \bigcup_{u=0}^{m-1} \mathcal{K}_u$ having non-vanishing proportion in the remaining total volume $\sum_{k' \in \mathcal{K} \setminus \bigcup_{u=0}^{m-1} \mathcal{K}_u} d_k^{(n)}$, and put them together to form $\mathcal{K}_m$.

By comparing the magnitude of the costs of users between different groups $\in \bigcup_{u=0}^{m} \mathcal{K}_u$, one can then find a suitable scaling factor sequence. For polynomial price functions, this can be done by comparing the degrees of the polynomials.

With the scaling factors, one can then apply the technique of [Wu et al. 2017 arXiv] to the marginal Γ_m. For polynomials, the marginal is scalable. Thus Theorem 1 holds for each marginal Γ_m.

Asymptotic decomposition

The proof is very long, see our online version [Wu et al. 2018 arXiv] for details.

However, the asymptotic decomposition is a very general idea. The decomposition can be done if there exists a suitable ordering of the price functions when the total volume approaches infinity..

We also demonstrated its generality by applying it to NCGs with regularly varying price functions, and proved that these more general games also have the same convergence of PoA.

References

1 R. Colini-Baldeschi, R. Cominetti, P. Mertikopoulos, and M. Scarsini. The asymptotic behav- ior of the price of anarchy. WINE 2017, N.R. Devanur and P. Lu (Eds), Lecture Notes in Computer Science 10674, pages 133145, 2017.

2 R. Colini-Baldeschi, R. Cominetti, and M. Scarsini. On the price of anarchy of highly congested nonatomic network games. SAGT 2016, M. Gairing and R. Savani (Eds), Lecture Notes in Computer Science 9928, pages 117128, 2016.

3 Z. Wu, R. H. Möhring, and Y. Chen. Selfishness need not be bad. arXiv:1712.07464v1[CS.GT], submitted, pages 151, 2017.

4 Z. Wu, R. H. Möhring, and D. Xu. Selfishness need not be bad: a general proof. arXiv:1805.16424v1[CS.GT], pages: 1-68, 2018.

Summary

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1. We formalized the concept of limit games and proposed an elementary theory for limit games, and applied the theory to the limit analysis of the PoA.

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2. We developed a general technique for the limit analysis of PoA, and completely proved the convergence of PoA for NCGs with polynomial price functions.
3. Our results imply that selfish routing is not the main cause of congestion when the total travel demand is large. Actually, it is the best choice in a "crowded" environment. This complements the partial theory of selfish behavior developed by Roughgarden & Tardos.

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3. Our results imply that selfish routing is not the main cause of congestion when the total travel demand is large. Actually, it is the best choice in a "crowded" environment. This complements the partial theory of selfish behavior developed by Roughgarden & Tardos.
4. Our NCG is very extensive. Our results apply also to contexts other than traffic. For instance, our results main imply that Free Market optimizes social manufacturing cost when we consider the NCG as a semi-macro model of social manufacturing system.

Congratulations to Prof. Du for his 70th birthday, and wish him good health and happy everyday!

Computing the PoA for Pigous's game

Consider the Pigou's game. Since there is only one user group (i.e., one O/D pair) with two strategies (i.e., the two paths), a feasible strategy profile f is a 2-dimensional vector (f_1, f_2) such that $f_1 + f_2 = T$ ($T \geq 1$ denotes the total user volume in thiss case), where f_1 denotes the volume of users adopting the upper strategy (path) and f_2 denotes the volume of users adopting the lower strategy (path).

Consider an NE profile $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$ and an SO profile $f^* = (f_1^*, f_2^*)$. Then

$$\tilde{f}_1^{\beta} = 1, \quad \tilde{f}_1 + \tilde{f}_2 = T.$$

Thus, $\tilde{f}_1 = 1$ and $\tilde{f}_2 = T - 1$. Therefore, $C(\tilde{f}) = \tilde{f}_1 \cdot \tilde{f}_1^{\beta} + \tilde{f}_2 \cdot 1 = T$.
Since f^* is an SO profile, it is then an NE profile w.r.t. price functions $x \cdot (x^{\beta})' + x^{\beta} = (\beta + 1)x^{\beta}$ (upper path) and $x \cdot 1' + 1 = 1$ (lower path). Similarly, one can then obtain that

$$(\beta + 1)(f_1^*)^{\beta} = 1, \quad f_1^* + f_2^* = T,$$

and thus $f_1^* = (\beta + 1)^{-1/\beta}$ and $f_2^* = T - (\beta + 1)^{-1/\beta}$. Then
$C(f^*) = f_1^* \cdot (f_1^*)^{\beta} + f_2^* \cdot 1 = (\beta + 1)^{-1-1/\beta} + T - (\beta + 1)^{-1/\beta}$. The PoA then follows.

Regularly varying functions

A function $g(\cdot)$ is called regularly varying if for each $x > 0$

$$\lim_{t\to\infty} \frac{g(tx)}{g(t)} = x^{\rho}$$

for some contant $\rho \in \mathbb{R}$. By Karamata's Characterization and Representation Theorem [Bingham 1987 Book], a regularly varying functions can be written as

$$g(x) = x^{\rho} \cdot \exp\left\{\eta(x) + \int_b^x \epsilon(z)/z \; dz\right\},$$

where $\eta(x)$ is a bounded function s.t. $\eta(x) \to \kappa$ as $x \to \infty$ for some constant $\kappa \in \mathbb{R}$, and $\epsilon(x) \to 0$ as $x \to \infty$, and $b > 0$ is a constant.